Ex Libris
OGDEN

THE LIBRARY
OF
THE UNIVERSITY
OF CALIFORNIA
LOS ANGELES

The Inokuty Technical Papers

Presentation Copy

From
the Celebration Committee

May, 1913

INOKUTY

TECHNICAL PAPERS

*TJ
7
I58t

PREFACE

PROF. Ariya Inokuty has occupied the chair of Engineering in the Engineering College of Tokyo Imperial University for upwards of twenty five years. With his energetic mind ever directed to progress in his profession, he is still actively employed in the duties of his Professorship, in the study of various subjects of interest to himself and his co-workers, and in promoting and carrying out researches in various departments of technical phenomena.

The graduates of the College educated under Prof. Inokuty now number many thousands. Most of them occupy responsible positions in Engineering circles. The labours of their former professor are always a matter of deep interest to them, and often of considerable value in the guidance of their own work.

On various occasions he has published essays relating to his researches; on other occasions he has expressed his views on matters connected with the actual as well as the ideal education of the Engineer. These essays and expressions of opinion and judgment are scattered throughout the Transactions of various Institutions and technical journals; they have had, in many cases, a lasting effect upon the development of the science and art of Mechanical Engineering.

To his very numerous pupils both past and present, the time appears to have arrived for a fitting recognition of the value of his services. The completion of the 25th year of his professorial work seems a suitable occasion for a special celebration. With the kind assistance of other friends of the Professor, these pupils

have already presented to the College a set of experimental centrifugal pumps with an electric motor to drive them. They now propose to take one step further.

To interest the general public, and make the Professor's work and researches available to as large a circle as possible, they propose—through the Celebration Committee appointed for the purpose—to compile his various contributions and essays, and reprint and publish them in a comprehensive volume. This volume will be called the "Inokuty Technical Papers." The contents will embrace his papers already published in Journals, Transactions of Institutions and elsewhere, and also other papers which have not yet publicly appeared. It is unnecessary to enlarge on the value that this volume will offer to its readers, representing as it does the outcome of his literary labours. It will bring within easy access what is now obtained only by some diligence and searching. It will be a souvenir to his friends and a guide to his fellow workers. The compilation of it is attractive, and the work itself should stand as a lasting monument to the Professor's great technical genius.

THE CELEBRATION COMMITTEE.

TOKYO,
March, 1913.

CONTENTS

PAGE

I. Volume of Certain Solids 1
[*Rigaku Kyokwai Zasshi.*]

II. Extension of Simpson's Rule 6
[*The Journal of the Engineering Society, Japan, No.* 75, 1888.]

III. Theory of Warming Buildings by Hot Water or Steam 16
[*The Journal of the Engineering Society, Japan, No.* 126, 1892.]

IV. Clearance in Steam Engine Cylinders 28
(In a letter to the Editor of 'The Engineer'.)
[*The Engineer, Vol. LXXX. August* 23, 1895.]

V. Strength of Wheel Teeth 38
[*The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. I, No.* 1, 1897.]

VI. Designing Pressure Blowers and Rotary Pumps 45
[*The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. I, No.* 1, 1897.]

VII. Effect of Inertia of Moving Parts of an Engine on Piston and Crank Pin Effort 53
[*Transaction of Japanese Institution of Naval Architects, No.* 1, *December*, 1897.]

VIII. Preliminary Experiment on the Strength of Bolt and Nut 64
[*The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. I, No.* 2, 1898.]

PAGE

IX. Strength of Long Tubes under External Pressure ... 69

[*The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. I, No.* 2, 1898.]

X. Interpolation 72

[*Hitherto unpublished.*]

XI. Theory of Fly Wheel as applied to Turbines and High Speed Engines 122

[*The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. IV, No.* 5, *June,* 1901.]

XII. Torsion Test of Cast-Iron 134

By A. Inokuty, *Kogakuhakushi, Kogakushi* and Fuji Tanaka, *Kogakushi*

[*Report of College of Engineering, Tokyo Imperial University, No, V, pp.* 7–9, *June,* 1903.]

XIII. Determination of Coefficients of Discharge from a Rectangular Notch 134

By A. Inokuty, *Kogakuhakushi, Kogakushi* and S. Utimaru, *Kogakushi*

[*Report of College of Engineering, Tokyo Imperial University, No. V, pp.* 4–6, *June,* 1903.]

XIV. Result of Tests of Vacuum Gauges 135

By A. Inokuty, *Kogakuhakushi, Kogakushi* and S. Utimaru, *Kogakushi*

[*Report of College of Engineering, Tokyo Imperial University, No. V, pp.* 1–3, *June,* 1903.]

XV. Strength of Flat Part of a Steam Boiler fitted with a Doubling Plate 136

[*The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. VI, No.* 9, *June,* 1903.]

PAGE

XVI. Sudden Stoppage of Water in a Pipe of Uniform Section 145

[*The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. VII, No.* 11, *September*, 1904.]

XVII. Sizes of Ready Made Engineering Articles 153

[*The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. VII, No.* 11, *September*, 1904.]

XVIII. Shearing Test of Timbers 163

By A. Inokuty, *Kogakuhakushi, Kogakushi* and Fuji Tanaka, *Kogakushi*

[*The Journal of the College of Engineering, Tokyo, Imperial University, Japan, Vol. II, No.* 3, *January*, 1905.]

XIX. Strain Energy of a Beam beyond the Elastic Limit ... 173

By A. Inokuty, *Kogakuhakushi, Kogakushi* and Fuji Tanaka, *Kogakushi*

[*The Journal of the College of Engineering, Tokyo Imperial University, Japan, Vol. II, No.* 3, *January*, 1905.]

XX. Universal Repetitive Bending Test 180

By A. Inokuty, *Kogakuhakushi, Kogakushi* and Fuji Tanaka, *Kogakushi*

[*The Journal of the College of Engineering, Tokyo Imperial University, Japan, Vol. II, No.* 3, *January*, 1905.]

XXI. Results of Tests of Pressure Gauges... 180

By A. Inokuty, *Kogakuhakushi, Kogakushi* and Fuji Tanaka, *Kogakushi*

[*The Journal of the College of Engineering, Tokyo Imperial University, Japan, Vol. II, No.* 3, *January*, 1905.]

PAGE

XXII. THEORY OF ORDINARY CENTRIFUGAL PUMPS AND OF A NEW CENTRIFUGAL PUMP HAVING DIVERGENT VORTEX CHAMBER PROVIDED WITH GUIDE VANES FOR PRODUCING FORCED VORTEX 181

[*The Journal of the College of Engineering, Tokyo Imperial University, Japan, Vol. II, No.* 4, *March* 1905. *The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. VIII, No.* 12, *May* 1905.]

XXIII. RESULTS OF TESTS OF A FORCED VORTEX CENTRIFUGAL PUMP 281

[*The Journal of the College of Engineering, Tokyo Imperial University, Japan, Vol. II, No.* 4, *March* 1905. *The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. VIII, No.* 12, *May* 1905.]

XXIV. FORMULÆ FOR HELICAL SPRINGS 297

[*The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. VIII, No.* 14, *October*, 1905.]

XXV. FLEXIBLE COUPLING 305

[*The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. VIII, No.* 15, *April*, 1906.]

XXVI. SIMILAR STRUCTURES WITH CORRESPONDING LOADS 307

[*The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. IX, No.* 17, *September*, 1907.]

XXVII. COLUMN OF UNIFORM STRENGTH... 320

[*Journal of the Engineering Society, Japan, Nos.* 300, 301, 302, 1907.]

XXVIII. STRENGTH OF FOUNDATION FOOTING FOR A HEAVILY LOADED STRUCTURE 343

[*The Journal of the Engineering Society, Japan, No.* 311, *November*, 1908.]

PAGE

XXIX. Strength of Curved Beams 347

By A. Inokuty, *Kogakuhakushi*, *Kogakushi* and K. G. Takemura, *Kogakushi*

[*The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. XIII, No.* 22, *May*, 1910.]

XXX. Method of Least Squares 365

[*Hitherto unpublished.*]

XXXI. Disposition of Boiler Feed Pipes... 427

[*The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. XIV, No.* 24, *February*, 1911.]

XXXII. The Relation between the Horse Power and the Weight of an Engine 429

[*The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. XV, No.* 26, *February* 1912.]

CONTENTS

OF THE

PAPERS IN JAPANESE

PAGE

I. Design of Water Wheels 1

[*Kogaku-soshi* (afterward—*The Journal of the Engineering Society, Japan*), *Nos.* 15, 16, 19, 1883.]

II. Japanese Water Wheels 27

[*The Journal of the Engineering Society, Japan, Nos.* 71, 76, 1887.]

III. Lift and Force Pumps 37

[*The Journal of the Engineering Society, Japan, No.* 119, *November*, 1891.]

IV. The Term Velocity in Japanese 56

[*The Journal of the Engineering Society, Japan, No.* 222, *February*, 1892.]

V. Report on Testing of Materials 58

By A. Inokuty, *M. E.*, B. Mano, *M. E., Kogakuhakushi* and S. Tanabe, *M. E., Kogakuhakushi*

[*Report of the Earthquake Investigation Committee, Japan, Japanese Edition No.* 2, *pp.* 155–162, *August*, 1894.]

VI. Second Report on Testing of Materials 59

By A. Inokuty, *M. E.*, B. Mano, *M. E., Kogakuhakushi* and S. Tanabe, *M. E., Kogakuhakushi*

[*Report of the Earthquake Investigation Committee, Japan, Japanese Edition No.* 3, *pp.* 15–26, *June*, 1895.]

PAGE

VII. NECESSITY OF ESTABLISHED RENDERING OF TECHNICAL TERMS IN JAPANESE 60

[*Read before the Meeting of the Members of the Society of Mechanical Engineers, Tokyo, Japan,* 1898; KIKAI-KÔGAKU-ZYUTUGO-SYÛ *Vol.* 1.]

VIII. ON FURNACE DRAUGHT... 64

[*The Journal of the Engineering Society, Japan, No.* 205, *February,* 1899.]

IX. NOTES ON PRACTICAL MATHEMATICS 79

[*A Report to the Technical Education Bureau in the Department of Education, October,* 1902.]

X. REPORT ON SHEARING TEST OF TIMBER 86

By A. INOKUTY, *M. E., Kogakuhakushi* and
FUJI TANAKA, *Kogakushi*

[*Report of the Earthquake Investigation Committee, Japan, Japanese Edition No.* 55, *September,* 1906.]

XI. ON MATHEMATICAL TRAINING OF ENGINEERS 108

[*Lecture at the Meeting of Masters of Technical Schools and Apprenticeship Schools in Japan. Report of the same, December,* 1911.]

ERRATA

(* Reference to line from bottom)

Page	Line		
144	table, last column, *read* $k=(1+m^3)$.		
146	6 *	*read*	$a+a\frac{dX}{dx}$.
183	9	*read*	$=\frac{p_3}{\rho}+\frac{s_3^2}{2g}=const.$
195	9 *	*read*	$\frac{w_1^2}{2g}$.
,,	5 *	*read*	$\frac{w_2^2}{2g}$.
200	table, 4th column, 2nd line from bottom, *read* ·310.		
249	13	*read*	ϕ_2'.
261	8	*read*	Blaine.
278	3	*read*	$\mu\frac{r_2}{r_3}\frac{r_3-r_2}{b_3\sin\alpha_3}\left(1+\frac{b_2\sin\alpha_2}{b_3\sin\alpha_3}\right)\frac{w_2^2}{2g}$.
285	5 *	*read*	drawn showing.
321	2	*read*	Gordon-Rankine.
324	3 *	*read*	double tee section.
332	table, last column, *read* $=\left(c-\frac{1}{2}m\right)^2+\frac{1}{12}m^2$.		
351	2 *	*read*	in his book.
357	1 *	*read*	line AB at.
360	3	*read*	perimeter.
372	9 *	*read*	that that an error.
381	4	*read*	lying within.
409	last determinant, *read* $\begin{vmatrix}1&0\\-1&2\end{vmatrix}$.		
,,	1 *	*read*	vz_3.
423	table, 3rd column, 3rd line from bottom, *read* 2·490.		
430	11	*read*	piston rod, connecting rod, and crank.
431	11	*read*	$p_o=\rho v^2/g$.

I.

Volume of Certain Solids.

[*Rigaku Kyokwai Zasshi.*]

Let a surface be generated by the motion of a straight line which is guided by the perimeters of any two closed plane curves $r_1=\phi_1(\theta)$ and $r_2=\phi_2(\theta)$ in parallel planes, and which moves in such a manner that it always lies in the plane containing the poles O_1 and O_2 of the said curves. Bounded by this surface and by the two plane curves, we obtain a solid somewhat like a frustum of any cone. It is proposed to find an expression for the volume of this solid.

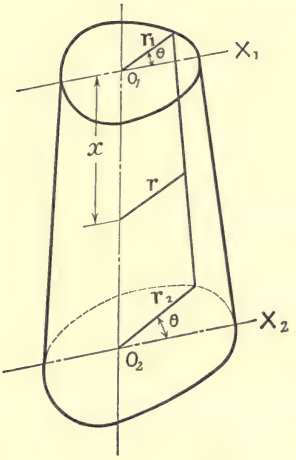
For simplicity we suppose the axis or the line $O_1\ O_2$ to be perpendicular to the two plane curves and employ the following notations:—

h = perpendicular distance between the two planes,

x = distance of any intermediate parallel section from the plane of $r_1=\phi_1(\theta)$,

r = radius vector of the section at distance x, the vectorial angle being θ.

Then $$r=(r_2-r_1)\frac{x}{h}+r_1,$$



and the area of any section is

$$A=\frac{1}{2}\int r^2\,d\theta$$

$$=\frac{1}{2}\int\left\{(r_2-r_1)\frac{x}{h}+r_1\right\}^2 d\theta$$

$$=\frac{x^2}{h^2}\frac{1}{2}\int r_2^2\,d\theta+\frac{2x}{h}\left(1-\frac{x}{h}\right)\frac{1}{2}\int r_1 r_2\,d\theta$$

$$+\left(1-\frac{x}{h}\right)^2\frac{1}{2}\int r_1^2\,d\theta,$$

which may be put thus:—

$$A=\frac{x^2}{h^2}A_2+\frac{2x}{h}\left(1-\frac{x}{h}\right)B+\left(1-\frac{x}{h}\right)^2 A_1.$$

Hence the volume of the solid is

$$V=\int_0^h A\,dx$$

$$=\int_0^h\left\{\frac{x^2}{h^2}A_2+\frac{2x}{h}\left(1-\frac{x}{h}\right)B+\left(1-\frac{x}{h}\right)^2 A_1\right\}dx$$

$$=\frac{1}{3}h(A_2+B+A_1), \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

where A_1 and A_2 are the areas of the two ends and B the following function:—

$$B=\frac{1}{2}\int\phi_1(\theta)\phi_2(\theta)\,d\theta.$$

It is easy to see that B is never greater than $\sqrt{A_1A_2}$.

If the figures forming the two ends of the solid are similar, the solid is a cone and we obtain the usual formula for it, thus

$$r_2=n\,r_1 \quad \text{or} \quad \phi_2(\theta)=n\times\phi_1(\theta).$$

$$\therefore \qquad B=\frac{1}{2}\int \phi_1(\theta)\, n\, \phi_1(\theta) d\theta = n\, A_1$$

$$= \sqrt{n^2 A_1^2} = \sqrt{A_1 \times n^2 A_1} = \sqrt{A_1 A_2}.$$

Hence $$V=\frac{1}{3}h(A_2+\sqrt{A_2 A_1}+A_1). \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

In order to find the numerical value of the function B, we take the following case:—A solid on a circular and a square ends. Putting d for the diameter of the circle and s for the side of the square, we have

$$r_1=\frac{1}{2}d,$$

and taking the initial line perpendicular to a side of the square, we have

$$r_2=\frac{\frac{1}{2}s}{\cos\theta}.$$

Now $$B=\frac{1}{2}\int r_1 r_2\, d\theta$$

$$=\frac{1}{2}\int \tfrac{1}{2}d \frac{\frac{1}{2}s}{\cos\theta} d\theta$$

$$=\frac{1}{8}s\,d \log\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right).$$

This integral is to be taken between the limits $\theta=\frac{\pi}{4}$ and $\theta=0$, and the result multiplied by 8.

$$\therefore \qquad B=s\,d \log\tan\frac{3\pi}{8}$$

$$=\sqrt{\frac{4}{\pi}}\log\tan\frac{3\pi}{8}\times\sqrt{s^2\frac{\pi}{4}d^2}$$

$$=\sqrt{\frac{4}{\pi}}\log\tan\frac{3\pi}{8}\sqrt{A_1 A_2}$$

$$=\cdot 99452\sqrt{A_1 A_2}.$$

Hence for this solid the volume is

$$V=\frac{1}{3}h(A_1+\cdot 99452\sqrt{A_1A_2}+A_2). \quad \ldots\ldots\ldots\ldots \quad (3)$$

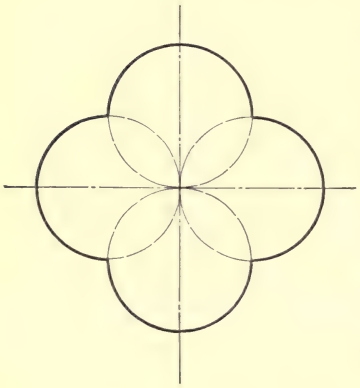
The coefficient of middle term is practically unity and may be left of consideration, and then the formula is just the same as for a frustum of any cone.

Again if we suppose the two guiding plane curves to be inverse to each other, we have $r_1r_2=\text{constant}=k^2$, say, and

$$B=\frac{1}{2}\int k^2 d\theta=\pi k^2,$$

so that the formula (1) reduces itself to

$$V=\frac{1}{3}h(A_1+\pi k^2+A_2). \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$



As an example of this class of solids, let one end be a square and the other its inverse, that is, four circles arranged as in the figure.

Putting d for the diameter of four equal circles and s for the side of the square, we have the volume of this solid

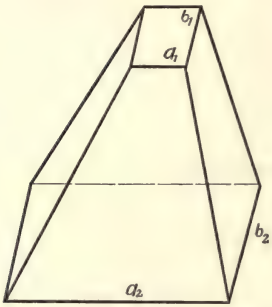
$$V=\frac{1}{3}h\left\{\left(\frac{\pi}{2}+1\right)d^2+\frac{1}{2}\pi\, s\, d+s^2\right\}$$

$$=\frac{1}{3}h\left\{A_1+\frac{\frac{1}{2}\pi}{\sqrt{(\frac{\pi}{2}+1)}}\sqrt{A_1A_2}+A_2\right\}$$

$$=\frac{1}{3}h(A_1+\cdot 97968\sqrt{A_1A_2}+A_2). \quad \ldots\ldots\ldots\ldots \quad (5)$$

Here again the coefficient of the middle term is approximately equal to unity.

It is to be remembered, however, that the solid here sketched does not come under the class of the solids whose volume we proposed to find. This solid here has two parallel rectangles for the ends, a_1 being parallel to a_2, and b_1 being parallel to b_2. The other faces are bounded by four planes. We will sketch out the method of finding an expression for its volume.



Putting a and b for the sides of any parallel section at distance x from top and using h for the perpendicular height, we have

$$\frac{a-a_1}{x}=\frac{a_2-a_1}{h} \quad \text{and} \quad \frac{b-b_1}{x}=\frac{b_2-b_1}{h}.$$

$$V=\int_0^h A\,dx=\int_0^h ab\,dx$$

$$=\frac{1}{3}h\left(a_1b_1+\frac{a_1b_2+a_2b_1}{2}+a_2b_2\right). \quad \ldots\ldots\ldots\ldots \quad (6)$$

This may be put thus:—

$$V=\frac{1}{3}h(A_1+c\sqrt{A_1A_2}+A_2),$$

where $$c=\frac{1}{2}\left(\sqrt{\frac{m_1}{m_2}}+\sqrt{\frac{m_2}{m_1}}\right),$$

and $a_1=b_1m_1$ and $a_2=b_2m_2$.

Suppose the two rectangular ends have their sides in the ratio of 1 : 2 and 2 : 1, then $c=1\frac{1}{4}$. This cannot be safely neglected.

II.

Extension of Simpson's Rule.

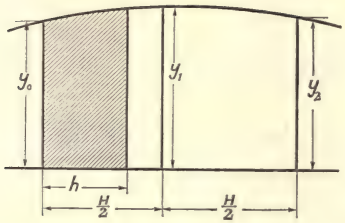
[*The Journal of the Engineering Society, Japan, No.* 75, 1888.]

The following results which are an extension of Simpson's Rule may be useful for finding the approximate values of areas, volumes, moments of inertia and other integrals, especially those relating to naval architecture. I have arrived at three distinct formulæ marked (A), (B), and (C); of these (A) would be most convenient and useful for practical applications, while (C) would probably be more exact but involves a greater amount of labour in multiplication, and lastly (B) might be very handy as well as legitimate in some cases but would not be accurate enough in other cases as will be seen presently. These formulae will be considered separately.

I. Let there be given three *equidistant* ordinates y_0, y_1, y_2, and the whole abscissa H between y_0, and y_2, of any plane curve. The problem is to find approximately the area of the figure bounded by the curve, the abscissa, the ordinate y_0, and by another ordinate corresponding to *any abscissa* h, which is a fraction of H.

Assuming the parabola of the second order for the curve, as in the Simpson's First Rule, we have

$$y = P + Qx + Rx^2.$$



The area required is expressed by

$$A=\int_0^h (P+Qx+Rx^2)dx$$

$$=Ph+\frac{1}{2}Qh^2+\frac{1}{3}Rh^3$$

$$=\frac{h}{6}(6P+3mQH+2m^2RH^2),\ldots(1)$$

where $m=h\div H$. Now when x is equal to 0, $\frac{1}{2}H$, H, we have the corresponding values of y equal to y_0, y_1, y_2 respectively.

Therefore $y_0=P$,

$$y_1=P+\tfrac{1}{2}QH+\tfrac{1}{4}RH^2,$$

and $$y_2=P+QH+RH^2.$$

Multiply these equations by $\frac{1}{6}hk_0$, $\frac{1}{6}hk_1$, $\frac{1}{6}hk_2$ respectively and add the results to equation (1); k_0, k_1, k_2 being indeterminate multipliers. We have then

$$A+\frac{1}{6}h(k_0y_0+k_1y_1+k_2y_2)$$

$$=\frac{1}{6}h\Big\{(6+k_0+k_1+k_2)P+(3m+\tfrac{1}{2}k_1+k_2)HQ$$

$$+(2m^2+\tfrac{1}{4}k_1+k_2)H^2R\Big\}. \quad \ldots\ldots\ldots (2)$$

Since we have thus introduced three arbitrary quantities, we are at liberty to introduce three conditions. Very well then let us equate the coefficients of P, Q, R in equation (2) each to zero; thus :—

$$\left.\begin{aligned} 6+k_0+k_1+k_2&=0\\ 3m+\tfrac{1}{2}k_1+k_2&=0\\ 2m^2+\tfrac{1}{4}k_1+k_2&=0\end{aligned}\right\}. \quad \ldots\ldots\ldots\ldots\ldots (3)$$

Hence the equation (2) becomes

$$A=-\frac{h}{6}(k_0y_0+k_1y_1+k_2y_2). \quad \ldots\ldots\ldots\ldots\ldots \quad (4)$$

It now only remains for us to determine the values of the three constants k_0, k_1, k_2 in terms of m from the equations (3). When they are found and substituted in (4) we obtain the following:—

$$A=\frac{h}{6}\left\{(6-9m+4m^2)y_0+4m(3-2m)y_1-m(3-4m)y_2\right\}, \ldots(A)$$

which is the formula required.

We will apply it to a few cases. In the first place suppose $m=\frac{1}{2}$, that is, suppose we wish to find the area corresponding to one half of the whole abscissa, we then obtain the well-known 5–8 rule,

$$A=\frac{h}{12}(5y_0+8y_1-y_2),$$

which is thus seen to be a particular case of the formula (A).

Suppose we want to find the area corresponding to one fourth of the whole abscissa, we have then

$$A=\frac{h}{12}(8y_0+5y_1-y_2),$$

which might be called 8–5 rule.

Again the area corresponding to three-fourths of the whole abscissa is

$$A=\frac{h}{4}(y_0+3y_1),$$

a singular result, the third ordinate disappearing altogether.

We give two other particular cases:—

when $m=\frac{1}{3}$,

$$A=\frac{h}{54}(31y_0+28y_1-5y_2);$$

when $m=\frac{2}{3}$,

$$A=\frac{h}{27}(8y_0+20y_1-y_2).$$

These results together with other cases are given in the following table of coefficients.

Formula (A).

$$A=\frac{h}{c}(c_0y_0+c_1y_1+c_2y_2).$$

Giving the area corresponding to any abscissa $h=m\ H$, having given three equidistant ordinates y_0, y_1, and y_2.

Table (A).

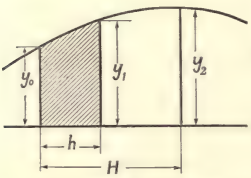
Giving the coefficients c_0, c_1, c_2, and the divisor c in the formula (A).

$m=\frac{h}{H}$	c_0	c_1	c_2	c
·1	5·14	1·12	−0·26	6
·2	4·35	2·08	−0·44	6
$\frac{1}{4}$	4·00	2·50	−0·50	6
	8	5	−1	12
·3	3·66	2·88	−0·54	6
$\frac{1}{3}$	3·44̇	3·11̇	−0·55̇	6
	31	28	−5	54
·4	3·04	3·52	−0·56	6
$\frac{1}{2}$	2·50	4·00	−0·50	6
	5	8	−1	12
·6	2·04	4·32	−0·36	6
$\frac{2}{3}$	1·77̇	4·44̇	−0·22̇	6
	8	20	−1	27
·7	1·66	4·48	−0·14	6
$\frac{3}{4}$	1·50	4·50	0	6
	1	3	0	4
·8	1·36	4·48	+0·16	6
·9	1·14	4·32	+0·54	6
1·0	1	4	+1	6
1·1	0·94	3·52	+1·54	6
1·2	0·96	2·88	+2·16	6
$1\frac{1}{4}$	1·00	2·50	+2·50	6
	2	5	+5	12

When it is more convenient to use integral coefficients and corresponding modified divisor, both sets of values are given in

the table. It may be noticed here that all the numbers in the table (except those with the signs of recurring decimal) are the exact values of the respective constants.

II. Instead of three equidistant ordinates, let us now suppose that y_1 is the ordinate corresponding to the abscissa h, as shown in the figure. Still using the parabola of the second order the integral for the area is the same as before, namely,



$$A=\frac{h}{6}(6P+3mQH+2m^2RH^2). \quad \ldots\ldots (1)$$

Now when x is equal to 0, h, H we have y equal to y_0, y_1, y_2 respectively.

$$\therefore \qquad y_0=P,$$

$$y_1=P+Qh+Rh^2$$

$$=P+mQH+m^2RH^2,$$

and $$y_2=P+QH+RH^2.$$

Proceeding in a manner similar to that of the preceding case we obtain the following three equations of conditions :—

$$\left.\begin{aligned} 6+k_0+k_1+k_2&=0\\ 3m+mk_1+k_2&=0\\ 2m^2+m^2k_1+k_2&=0 \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots\ldots (3)$$

together with the same equation as (4) in the preceding case for the area. When the constants are found and substituted in (4), there results

$$A=\frac{h}{6}\left\{(3-m)y_0+\left(3+\frac{m}{1-m}\right)y_1-\left(\frac{m^2}{1-m}\right)y_2\right\}, \quad \ldots\ldots\ldots (B)$$

which is the formula required.

By making the ratio m in this equation equal to one half we obtain the 5-8 rule, as it ought to be the case.

When $m=\frac{1}{4}$, $A=\frac{h}{72}(33y_0+40y_1-y_2)$.

When $m=\frac{3}{4}$, $A=\frac{h}{8}(3y_0+8y_1-3y_2)$.

When $m=\frac{1}{3}$, $A=\frac{h}{36}(16y_0+21y_1-y_2)$.

When $m=\frac{2}{3}$, $A=\frac{h}{18}(7y_0+15y_1-4y_2)$.

It is easy to see that the formula (B) would be more accurate than formula (A) so long as h is less than $\frac{1}{2}H$, while on the other hand (A) would be more accurate when h is greater than $\frac{1}{2}H$; when h approaches very near to the value H, the formula (B) would be almost useless for practical applications. The following table might be of some use in some calculations.

Formula (B).

$$A=\frac{h}{c}(c_0y_0+c_1y_1+c_2y_2).$$

Giving the area corresponding to any abscissa $h=mH$, having given the extreme ordinates y_0, y_2 and the ordinate y_1 corresponding to h.

Table (B).

Giving the coefficients c_0, c_1, c_2, and the divisor c in the formula (B).

$m=\frac{h}{H}$	c_0	c_1	c_2	c
·1	2·90	3·11	−0·011	6
·2	2·80	3·25	−0·050	6
$\frac{1}{4}$	2·75	3·33	−0·083	6
	33	40	−1	72
·3	2·70	3·43	−0·129	6
$\frac{1}{3}$	2·67	3·50	−0·167	6
	16	21	−1	36
·4	2·60	3·67	−0·267	6
$\frac{1}{2}$	2·50	4·00	−0·500	6
	5	8	−1	12
·6	2·40	4·50	−0·900	6
$\frac{2}{3}$	2·33	5	−1·333	6
	7	15	−4	18
·7	2·30	5·33	−1·633	6
$\frac{3}{4}$	2·25	6·00	−2·250	6
	3	8	−3	8

III. Let us now suppose that there are four equidistant ordinates y_0, y_1, y_2, y_3 and the whole abscissa H and that we want to find an approximate value of the area up to any abscissa h. Assuming the parabola of the third order for the curve as in the Simpson's Second Rule,

$$y=P+Qx+Rx^2+Sx^3,$$

the expression for the area is

$$A=\int_0^h (P+Qx+Rx^2+Sx^3)dx$$

$$=Ph+\tfrac{1}{2}Qh^2+\tfrac{1}{3}Rh^3+\tfrac{1}{4}Sh^4$$

$$=\frac{h}{12}(12P+6mQH+4m^2RH^2+3m^3SH^3). \quad \ldots\ldots\ldots\ldots (1)$$

When $x=0$, $\frac{1}{3}H$, $\frac{2}{3}H$, H, we have the values of y equal to y_0, y_1, y_2, y_3 respectively. There will then be four equations for determining the four constants P, Q, R, S.

Proceeding just in the same way as before we obtain the following :—

$$A=\frac{h}{8}\Big\{(4-3m)(3m^2-4m+2)y_0+3m(9m^2-20m+12)y_1$$

$$-3m(9m^2-16m+6)y_2+m(2-3m)^2y_3\Big\}, \quad \ldots\ldots\ldots\ldots (C)$$

which is the formula required.

When m is unity this formula becomes Simpson's Second Rule.

When $m=\frac{1}{3}$, $\quad A=\frac{h}{24}(9y_0+19y_1-5y_2+y_3)$.

When $m=\frac{1}{2}$, $\quad A=\frac{h}{64}(15y_0+51y_1-3y_2+y_3)$.

When $m=\frac{2}{3}$, $\quad A=\frac{h}{12}(y_0+4y_1+y_2)$.

Formula (C).

$$A=\frac{h}{c}(c_0y_0+c_1y_1+c_2y_2+c_3y_3).$$

Giving the area corresponding to any abscissa $h=mH$, having given four equidistant ordinates y_0, y_1, y_2, and y_3.

Table (C).

Giving the coefficients c_0, c_1, c_2, c_3, and the divisor c in the formula (C).

$m=\frac{h}{H}$	c_0	c_1	c_2	c_3	c
·1	6·031	3·027	−1·347	0·289	8
·2	4·488	5·016	−1·896	0·392	8
·25	3·859	5·672	−1·922	0·391	8
·3	3·317	6·129	−1·809	0·363	8
$\frac{1}{3}$	3·000	6·333	−1·667	0·333	8
	9	19	−5	1	24
·4	2·464	6·528	−1·248	0·256	8
$\frac{1}{2}$	1·875	6·375	−0·375	0·125	8
	15	51	−3	1	64
·6	1·496	5·832	+0·648	0·024	8
$\frac{2}{3}$	1·333	5·333	+1·333	0	8
	1	4	+1	0	12
·7	1·273	5·061	+1·659	0·007	8
·75	1·203	4·641	+2·109	0·047	8
·8	1·152	4·224	+2·496	0·128	8
·9	1·079	3·483	+2·997	0·441	8
1·0	1	3	+3	1	8

III.

Theory of Warming Buildings by Hot Water or Steam.

[*The Journal of the Engineering Society, Japan, No.* 126, 1892.]

The rationale of the apparatus for warming buildings by hot water or steam will not be out of place in this journal. Some of the following formulæ and the methods of calculation relating to the apparatus are believed to be new so far as the writer is aware and they are applicable to any of the systems that are in vogue at present.

Let q=the quantity of heat in British thermal units given out per minute by a given system of pipes,

l=the length of pipes in ft.,

d=the external diameter of pipes in inches,

d_0=the internal diameter of pipes in inches,

t=the temperature of hot water or steam in the pipe in degrees Fah.,

t'=the temperature of the air surrounding the pipe, or the temperature of the air in the room after it has been warmed, in degrees Fah.

Then $$q=k\frac{\pi d}{12}l(t-t'),$$

where k is the rate of combined radiation, conduction, and convection per square foot of pipe surface per degree of difference of temperature per minute. To determine the value of k we quote from D. K. Clark's Rules, Tables, and Data p. 481. "Mr. Hood estimates, from experiments made by Tredgold, that the water contained in an iron pipe of 4 inch diameter internally and $4\frac{1}{2}$ externally loses 0·851° F. of heat per minute when the excess of its temperature is 125° F. above that of the surrounding air,....." From these data we have

$$\left\{62{\cdot}4l\left(\frac{\frac{\pi}{4}\times 4^2}{144}\right)+{\cdot}115\times 480l\left(\frac{\frac{\pi}{4}\times 4\frac{1}{2}^2-\frac{\pi}{4}\times 4^2}{144}\right)\right\}\times 0{\cdot}851$$
$$=k\frac{\pi\times 4\frac{1}{2}}{12}\times l\times 125,$$

where the number 0.115 is the specific heat of iron; by calculation the value of k is found to be 0·0389 British thermal unit per square foot of pipe surface per degree of difference of temperature per minute.

Thus we have $q=0{\cdot}0389\frac{\pi d}{12}l(t-t')$ heat units per min. ...(1)

$q=2{\cdot}334\frac{\pi d}{12}l(t-t')$ heat units per hour....(1′)

The value of k may be calculated from other data. In D. K. Clark's Rules, Tables and Data are given (p. 472—p. 477) the results of various experiments made by different authorities on the condensation of steam in iron pipes. The average value of the amount of condensation is 0·00235 lb. of steam at 20 lbs. per square inch absolute pressure, per square foot of pipe surface, per degree of difference of temperature, per hour. Now the latent heat of evaporation of steam at the pressure above stated is 954·8 British thermal units, and consequently the value of k ought to be

$$k=\frac{0\cdot00235\times954\cdot8}{60} \quad =0\cdot0374 \text{ heat unit per minute,}$$

$$=2\cdot244 \text{ heat units per hour.}$$

Now suppose that the volume of air to be warmed per minute is V cubic feet estimated at the temperature of $32°F.$ from $t''F.$ to $t'F.$, i.e., from the temperature of the air outside the room to that in the room. The volume of one lb. of air at $32°F.$ occupies 12·39 cubic ft. and the specific heat of air is 0·2377. We therefore have from the preceding formula

$$0\cdot2377\times\frac{V}{12.39}(t'-t'')=0\cdot0389\times\frac{\pi d}{12}l(t-t'),$$

from which we obtain $$l=1\cdot90\frac{V(t'-t'')}{d(t-t')}=\frac{2V(t'-t'')}{d(t-t')}, \text{ say. ... (2)}$$

If calculated according to Mr. Hood's estimates who took the specific heat of air at 0·2767, the coefficient in the above formula would become 2·24 which is considered rather too large.

The temperature of hot water in the pipe may be about $200°F.$ It is said that in habitable rooms each square foot of glass may be considered as equivalent to an additional quantity of air 1¼ cubic ft. to be warmed per minute. For conservatories, forcing houses, and like buildings, the quantity of air to be warmed is 1¼ cubic ft. per square foot of glass per minute. The radiation of heat from frames and sashes made of metal is as great as from glass. When pipes are placed in trenches covered with grating, the loss of heat as estimated by Mr. Hood amounts to 5 to 7 per cent.

A man makes about 17 respirations per minute each of 40 cubic inches, that is, $17\times40\times60\div1728=23\cdot6$ cubic ft. per hour. An ordinary gas burner consumes about 5 cubic ft. of gas per

hour, and requires for combustion 12 cubic ft. of air per cubic ft. of gas, that is, 60 cubic ft. of air per hour for each gas burner; each pound of candles or oil burnt requires 160 cubic ft. of air for combustion.

In some book the quantity of air required for the proper ventilation of various buildings is stated to be as follows:—

	Cubic ft. per head per hour.	Cubic ft. per head per minute.
For apartments with healthy occupants	300*	5*
For apartments with sick occupants	1200	20
For prisons and workhouses	350*	5·8*
For churches and assembly rooms	550*	9·2*
For hospitals ordinary, and barracks	2200	36·7
For hospitals for infectious diseases	4500	75

The same book says that the space provided for each bed in the wards of oridinary hospitals, should not be less than 1800 cubic ft. and in hospitals for infectious diseases not less than 2500 cubic ft. The space provided in dwelling houses should not be less than 300 cubic ft. for each person in a room, whether children or adults.

According to modern investigations on ventilation, those quantities in the above table which are marked with asterisks * are considered far too small. The following is an extract from Encyclopædia Britanica, Article, Ventilation:—

"Parkes advises a supply of 2000 cubic ft. of air per hour (=33 cubic ft. per minute) for persons in health and 3000 or 4000 cubic ft. (=50 or 67 cubic ft. per minute) for sick persons. The English Barracks Improvement Commissioners require that

the supply be not less than 1200 cubic ft. (=20 cubic ft. per minute) per man per hour. Pettenkofer recommends 2100 cubic ft. (=35 cubic ft. per minute) and Morin considers that the following allowances are not too high :—

	Cubic ft. per head per hour.	Cubic ft. per head per minute.
Hospitals (ordinary)	2000 to 2400	33 to 40
„ (epidemic)	5000	83
Workshops (ordinary)	2000	33
„ (unhealthy trades)	3500	58
Prisons	1700	28
Theatres	1400 to 1700	23 to 28
Meeting halls	1000 to 2000	17 to 33
Schools (per child)	400 to 500	6.7 to 8.3
„ (per adult)	800 to 1000	13 to 17"

"A general idea of the cubic capacity per inmate allowed by law or custom in certain cases is given in the following table :—

	Cubic feet
Hospitals	1200 (& upwards)
Middle class houses	1000 (& upwards)
Barracks	600
Good secondary schools	500
London board schools	130
Workhouse dormitories	300
London lodging houses	240
One roomed houses*	212

* Mean of 29 measurements."

The simplest way of estimating the volume of air to be warmed is to calculate beforehand the number of times the air in a room ought to be renewed per hour. Dividing the whole contents of the room by the cubic capacity per inmate, we obtain the number of inmates; this number multiplied by the amount of ventilation in cubic ft. per head per hour gives the quantity of air to be warmed per hour; this product divided by the contents of the room gives the number of times the air in the room ought to be renewed per hour. The whole arithmetic is equivalent to simply dividing the amount of ventilation per hour by the cubic capacity per inmate. Thus suppose that for an ordinary dwelling house we allow 2100 cubic ft. of air per head per hour with 800 cubic ft. of room space allotted to each person in the room, then $2100 \div 800 = 2.6$ times per hour, that is, the air in the room must be wholly changed every 23 minutes. Again for a meeting hall let the ventilation be 1200 cubic ft. per head per hour with a room space of 200 cubic ft. per head, then the renewal of the air is $1200 \div 200 = 6$ times per hour. Again for a certain workshop the ventilation is 3000 cubic ft. per head per hour with an allowance of 350 cubic ft. of room space per head, then the renewal is 8.6 times per hour. It will be seen that in general where a large amount of ventilation is requisite for each person as in hospitals, the cubic capacity allowed per inmate must also be large, so that the number of times the air in the room is to be changed per hour may not be so high as might appear at first sight. While on the other hand in places where many people are crowded as in theatres, meeting halls, &c., the occupants are healthy persons and the amount of ventilation for each person need not be so free and abundant as in the first case, so that here also the number of times the air is to be changed per

hour is not very high. The following table gives the values of this number for various amounts of ventilation and different cubic capacities allowed per inmate:—

Ventilation in cubic ft. per head per hour.	Number of times the air in a room is to be changed per hour, for cubic ft. capacity per inmate =								
	200	250	300	350	400	500	600	700	800
1200	6·0	4·8	4·0	3·4	3·0	2·4	2·0	1·7	1·5
1500	7·5	6·0	5·0	4·3	3·75	3·0	2·5	2·1	1·9
1800	9·0	7·2	6·0	5·1	4·5	3·6	3·0	2·6	2·25
2100	10·5	8·4	7·0	6·0	5·25	4·2	3·5	3·0	2·6
2400	12·0	9·6	8·0	6·9	6·0	4·8	4·0	3·4	3·0
2700	13·5	10·8	9·0	7·7	6·75	5·4	4·5	3·9	3·4
3000	15·0	12·0	10·0	8·6	7·5	6·0	5·0	4·3	3·75
3300	16·5	13·2	11·0	9·4	8·25	6·6	5·5	4·7	4·1
3600	18·0	14·4	12·0	10·3	9·0	7·2	6·0	5·1	4·5

The use of the above table is this: multiply the cubic contents of the building to be warmed by one of the numbers in the table, properly chosen, and divide the product by 60; the result is the value of V in the formula (2).

To find the value of V, we may also make use of the following table which gives the volume of air in cubic ft. to be warmed per minute per 1000 cubic ft. of room space for various amounts of ventilation per head and different cubic capacities allowed per inmate:—

Ventilation in cubic ft. per head per hour.	Volume of air in cubic ft. to be warmed per minute per 1000 cubic ft. of room space, for cubic ft. capacity per inmate=								
	200	250	300	350	400	500	600	700	800
1200	100	80	67	57	50	40	33	29	25
1500	125	100	83	71	63	50	42	36	31
1800	150	120	100	86	75	60	50	43	37.5
2100	175	140	117	100	88	70	58	50	44
2400	200	160	133	114	100	80	67	57	50
2700	225	180	150	129	113	90	75	64	56
3000	250	200	167	143	125	100	83	71	62.5
3300	275	220	183	157	138	110	92	79	69
3600	300	240	200	171	150	120	100	86	75

Multiply the cubic contents of the building to be warmed by one of the numbers in the above table, properly chosen, and divide the product by 1000; the result is the value of V in formula (2).

We quote from D. K. Clark's Rules, Tables and Data p. 483 approximate rules for the length of 4 inch pipe required to warm any building for each 1000 cubic ft. of room space.

[Table.

" Building	Length of 4" pipe per 1000 cub. ft. of room space.	Temperature in Fah. maintained in the room.	Remarks.
Churches and large public rooms	5 ft.	55° *F.*	In very cold weather. If the air is regularly changed, from 50 to 70 per cent more pipe is required.
Dwelling rooms	12	65	
Ditto	14	70	
Halls, shops, waiting rooms, &c.	10	55	
Ditto	12	60	
Work-rooms, manufactories, &c.	6	50 to 55	
Ditto	8	60	
Schools and lecture rooms	6 to 7	55 to 58	
Drying rooms for wet linen, &c., when empty	150 to 180	120	
Ditto, when filled	,, ,,	80	
Drying rooms for curing bacon, drying paper, leather, hides	20	70	
Green houses & conservatories	35	55	In coldest weather.
Graperies & store-houses	45	65 to 70	Ditto
Ditto	50	70 to 75	Ditto
Pineries, hot houses, and cucumber pits	55	80	Ditto

These numbers for the length of 4 inch pipe are only suitable for buildings on the usual plan and of ordinary proportions."

Having determined the principal dimensions of a system of pipes for warming any building, the next step is to calculate the quantity of coal to be burnt per hour and then having decided on a given type of boiler for generating hot water or steam, we are to calculate its dimensions. The quantity of heat given out by a pipe d inches in diameter and l ft. long containing hot water or steam at t *F.* is given by the formula (1), namely,

$$q = 0{\cdot}0389 \times \frac{\pi d}{12} l(t - t') \text{ British thermal units per minute,}$$

where t' is the temperature of the air maintained in the room. The above may be put in lbs. of water evaporated per hour from and at 212° *F.*, thus:—

$$W = \frac{60 \times 0{\cdot}0389\pi}{966 \times 12} \times dl(t - t') = \frac{dl(t - t')}{1580} \text{ lbs.} \quad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

Let E' be the actual evaporative power of one lb. of coal, that is, the number of lbs. of water which is actually evaporated from and at 212° *F.* by one lb. of coal burnt. The value of E' varies according to the ratio of heating surface to the grate area, the type of boiler used, the method of producing the draught, &c., and it ranges from about 9 to 6 lbs. Now W divided by E' gives the quantity of coal required per hour, without any allowance for the loss of heat by evaporation to the open air, the loss by leakage, the loss of heat from boiler surface by radiation, &c. The rest of the calculation is just the same as for a steam engine boiler. Suppose for example we take d at 4″ and l at 1200 ft., the temperatures of the hot water in the pipe and that of the air in the room being 206° *F.* and 60° *F.* respectively.

Here $W = \frac{4 \times 1200(206 - 60)}{1580} = 444$ lbs. of water evaporated per hour.

Assuming the value of E' at 7 lbs. we have

$W \div E' = 444 \div 7 = 63$ lbs. of coal to be burnt per hour.

We will now investigate the relation between the rate of cooling and the velocity of circulation of hot water in a hot water apparatus. Suppose that a boiler can supply hot water at constant temperature t_0 *F.* through a system of pipes and that the air surrounding the pipe is kept at a constant temperature $t'F$.

by means of well regulated ventilation, then the temperature of the hot water in the pipe will gradually fall as we go away from the boiler. Now our problem is the following:—In order that the temperature of the hot water in the pipe at a distance x ft. from the boiler may be t *F.*, what must be the velocity of circulation?

Let v=velocity of hot water in ft. per minute,
A=sectional area of pipe in square ft.,
S=perimeter of pipe in ft.,
t_0=temperature Fah. in the boiler,
t=temperature Fah. of hot water at distance x ft. from the boiler,
t'=temperature Fah. of the air surrounding the pipe,
k=as before, the rate of combined conduction, convection and radiation per square ft. of pipe surface per degree of difference of temperature per minute,
w=weight in lbs. per cubic ft. of hot water,
c=specific heat of hot water.

Then the quantity of heat given out from the surface of a very short length of the pipe δx will be

$$-\delta Q = k(\delta x S)(t-t').$$

This amount of heat must be accounted for in the fall of temperature δt and hence we have also

$$\delta Q = Avwc\,\delta t.$$

Equating these two expressions for δQ, we have

$$Avwc(-\delta t) = k\delta x S(t-t'). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Hence $$v=\frac{k(t-t')}{wc\left(\frac{A}{S}\right)\left(-\frac{\delta t}{\delta x}\right)} \quad \text{or} \quad v=\frac{k(t-t')}{wcm\tau}, \quad \ldots\ldots\ldots (5)$$

where m=the hydraulic mean depth of the cross section of the pipe in ft.,

τ=the rate of fall of temperature per foot length of the pipe.

Again (4) may be written thus $$wcmv\frac{\delta t}{t-t'}=-k\delta x.$$

Integrating, we obtain $$wcmv\log(t-t')=-kx+\text{const.}$$

But when $t=t_0$=the temperature in the boiler, the distance x from it is zero. This determines the value of the constant, and we obtain

$$v=\frac{kx}{wcm\log\left(\frac{t_0-t'}{t-t'}\right)}. \quad \ldots\ldots\ldots (6)$$

(Not finished.)

IV.

Clearance in Steam Engine Cylinders.

(In a letter to the Editor of 'The Engineer'.)

[*The Engineer, Vol. LXXX. August* 23, 1895.]

Sir,

The fact that there always exists a certain amount of loss of work due to clearance so long as the terminal pressure is higher than the back pressure before compression is now undeniable, and it can be shown that there is a particular value of compression for which this loss is the least possible.* The following calculations relating to this subject may be of some interest. Two cases present themselves.

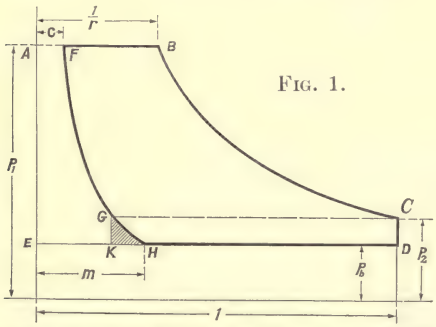


FIG. 1.

Case I.—Here the amount of compression is supposed to be sufficient to create a negative pressure just equal to the initial pressure. Referring to Fig. 1, let *BC* be the expansion curve, *FGH* the com-

* After having written this letter, the writer has just been able to learn that the existence of such a minimum loss had been shown many years ago.

pression curve, ED the back-pressure line, and AF the clearance space. From the end C of the expansion curve, draw horizontal line meeting the compression curve at the point G, we have then the triangular area GHK, representing the loss of work due to clearance and the sudden drop of pressure at CD. Assuming the hyperbolic law for the compression curve, and using the notations indicated in the figure, the area GHK has the following expression :—

$$cp_1\left\{\log r' - (1+\log r) - \frac{r}{r'}\right\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

where $r' = m \div c = p_1 \div p_b$. This represents the loss of work per $\left(\frac{1}{r} - c\right)$ cubic feet of steam at the initial pressure p_1. Now with the same initial and back pressure, and the same (real) ratio of expansion, the amount of work that could have been done, were there no clearance, is—

$$(1 - c\,r)\frac{p_1}{r}\left\{1 + \log r - \frac{r}{r'}\right\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Dividing (1) by (2), there results—

$$L = \frac{c\,r}{1 - c\,r}\left\{\frac{\log r'}{1 + \log r - \frac{r}{r'}} - 1\right\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (A)$$

which is then the loss of work due to clearance when the compression is sufficient to produce a pressure equal to that of the initial steam, and is expressed as a fraction of the total work which could have been done under an exactly similar condition without clearance. The above expression vanishes when c is zero; and it vanishes also when $p_1 \div p_b = r$, that is, when the terminal pressure is equal to the back pressure, as it ought to be the case. In the Table I, the terminal pressure p_2 is assumed

to be 25 lb. absolute, the back pressure p_b 17 lb. absolute. The volume m of the cushion steam at the instant when the compression commences, and also the volume of clearance, are expressed as fractions of the total cylinder volume. These may be converted into fractions of the piston displacement by putting $c \div (1-c)$ in place of c and $m \div (1-c)$ in place of m.

Table I.

c	$r=3{\cdot}4$ $p_1=85$ lbs. $p_b=17$ lbs.		$r=4$ $p_1=100$ lbs. $p_b=17$ lbs.		$r=4{\cdot}6$ $p_1=115$ lbs. $p_b=17$ lbs.	
	m	L	m	L	m	L
·03	·15	·005	·18	·005	·20	·006
·05	·25	·009	·29	·010	·34	·011
·07	·35	·013	·41	·016	·47	·017
·10	·50	·022	·59	·026	·68	·030
·14	·70	·039	·82	·049	·95	·064

With condensing engines it is usually not possible to obtain the compression pressure high enough to equal the initial pressure, unless either the clearance space is very small or the back pressure is comparatively high, or the initial pressure is very low.

Case II.—This is the more general case in which the compression is incomplete, as in most actual engines. We still assume the hyperbolic law for both the expansion and compression curve. It must be noticed that when the compression is incomplete, the steam from the boiler rushes into the partially vacuous clearance space at—or rather a few moments before—the beginning of each stroke without doing any work, and that this

process has the effect of making the steam a little drier. This is altogether neglected in the following calculation, so that all the figures for the loss of work are exaggerated more or less, and in reality the amount of compression required for the minimum loss of work would be much smaller than the calculation shows. Referring to Fig. 2, let *GHK* be the compression curve

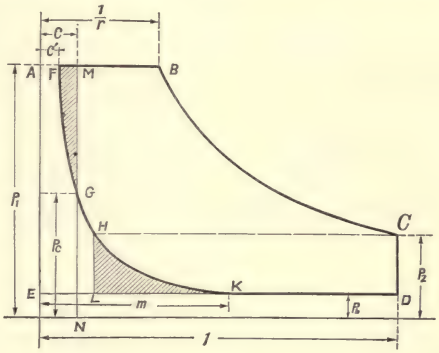


FIG. 2.

intersecting the vertical line *MG*, drawn at the end of the clearance space, in the point *G*. If this compression curve be continued till it meets the admission line at *F*, we obtain a narrow triangular area *FGM*. Now the loss of work due to clearance is the sum of the areas *HKL* and *FGM*, and the consumption of steam at pressure p_1 is $\frac{1}{r}-c'$ cubic feet per stroke, because $c'=AF$ is the volume to which the cushion steam must be compressed in order to obtain a negative pressure equal to that of the initial steam, and the compression curve as well as the expansion curve for the next double stroke would be the same, whether the compression be carried up to the point *F* or only to the point *G*. To obtain an expression for the loss of work, we first notice the following:—

$$c'=m\frac{p_b}{p_1},\quad \frac{1}{r}-c'=\frac{1}{r}-m\frac{p_b}{p_1},\quad r'=\frac{p_1}{p_b},$$

and

$$\frac{c}{c'}=\frac{c}{m}\cdot\frac{p_1}{p_b}=r'\frac{c}{m}.$$

The area for the triangular piece *HKL* will be just the same as in Case I, except that we have here to put c' in place of c, thus—

$$c' p_1 \left\{ \log r' - \left(1 + \log r - \frac{r}{r'} \right) \right\}$$

that is,

$$m p_b \left\{ \log r' - \left(1 + \log r - \frac{r}{r'} \right) \right\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

The expression for the narrow triangular area *FGM* is—

$$p_1 c - c \times \frac{p_1}{\left(\frac{c}{c'} \right)} \left(1 + \log \frac{c}{c'} \right)$$

that is,

$$p_1 c - m p_b \left(1 + \log r' - \log \frac{m}{c} \right). \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Adding (3) and (4) together, we have the loss of work per $\left(\frac{1}{r} - c' \right)$ cubic feet of steam at p_1, thus—

$$p_b m \log \frac{m}{c} + (p_1 c - p_b m) - p_b m \left(1 + \log r - \frac{r}{r'} \right). \quad \ldots\ldots \quad (5)$$

But the amount of work which could have been done by $\left(\frac{1}{r} - c' \right)$ cubic feet of steam under an exactly similar condition, but without clearance, is—

$$(1 - c'r) \frac{p_1}{r} \left\{ 1 + \log r - \frac{r}{r'} \right\}$$

that is,

$$\left(1 - m \frac{r}{r'} \right) \frac{p_1}{r} \left(1 + \log r - \frac{r}{r'} \right). \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

Hence, the loss of work expressed as a fraction becomes—

$$L=\frac{p_b m\left\{\log\frac{m}{c}+\frac{p_1 c}{p_b m}-1-\left(1+\log r-\frac{r}{r'}\right)\right\}}{\left(1-m\frac{r}{r'}\right)\frac{p_1}{r}\left(1+\log r-\frac{r}{r'}\right)}$$

$$=\frac{m\frac{r}{r'}}{1-m\frac{r}{r'}}\left\{\frac{\log\frac{m}{c}+\frac{cr'}{m}-1}{1+\log r-\frac{r}{r'}}-1\right\}. \quad \ldots\ldots\ldots \text{(B)}$$

This loss of work becomes the least possible when the volume m of the cushion steam has the value given by the equation—

$$\log\frac{m}{cr}=1-r\left(c+\frac{1-m}{r'}\right),$$

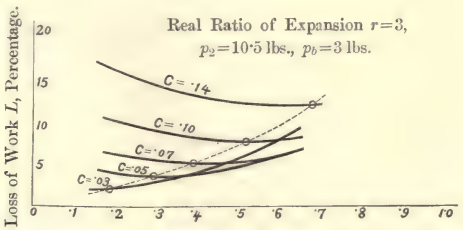
which has been found by applying the usual rule of calculus to the expression (B). The minimum loss of work then becomes—

$$L_{min.}=\frac{r\left(c-\frac{m}{r'}\right)}{1+\log r-\frac{r}{r'}},$$

which may be put in the following form :—

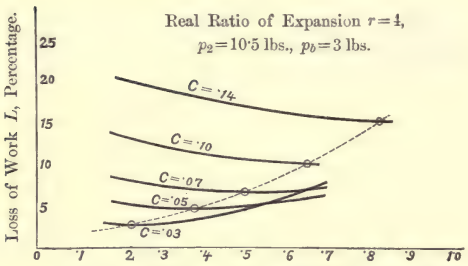
$$L_{min.}=\frac{(p_1-p_c)c}{\frac{p_1}{r}(1+\log r)-p_b}=\frac{\text{rectangle } AG}{ABCDE},$$

where p_c stands for the terminal compression pressure represented by the line GN in Fig. 2.



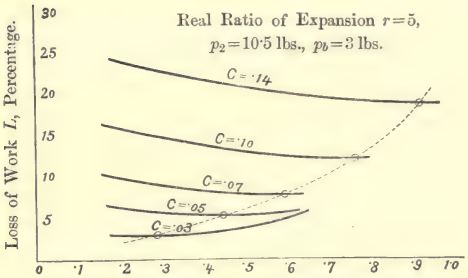
Compression Volume m in Fractions of Total Cylinder Volume.
Dotted Curve is the Locus of Minimum Loss.

FIG. 3.



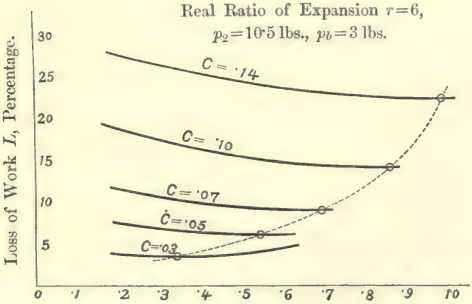
Compression Volume m in Fractions of Total Cylinder Volume.
Dotted Curve is the Locus of Minimum Loss.

FIG. 4.



Compression Volume m in Fractions of Total Cylinder Volume.
Dotted Curve is the Locus of Minimum Loss.

FIG. 5.



Compression Volume m in Fractions of Total Cylinder Volume.
Dotted Curve is the Locus of Minimum Loss.

FIG. 6.

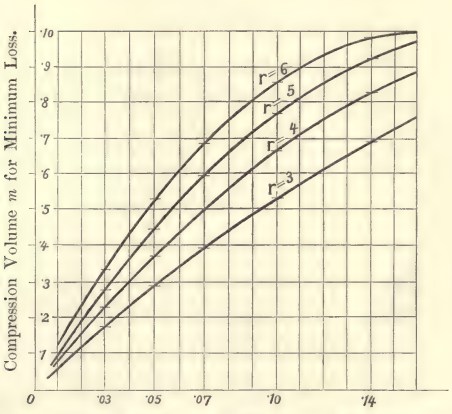


FIG. 7.

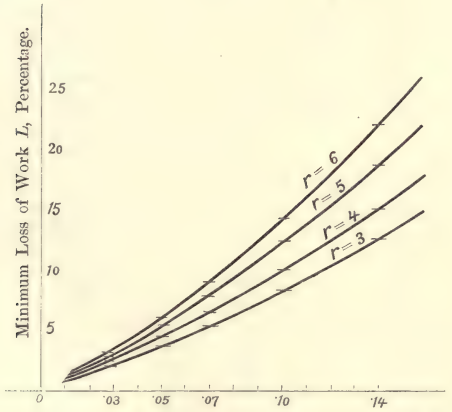


FIG. 8.

In the Table II. are given the values of the loss of work for various proportions of clearance space, compression volume, and the ratio of expansion. The table also gives the least loss

Table II.

$p_2 = 10{\cdot}5$ lbs., $p_b = 3$ lbs.

c	$r=3$ $p_1=rp_2=31{\cdot}5$ lbs.		$r=4$ $p_1=rp_2=42$ lbs.		$r=5$ $p_1=rp_2=52{\cdot}5$ lbs.		$r=6$ $p_1=rp_2=63$ lbs.	
	m.	L.	m.	L.	m.	L.	m.	L.
.03	·177	*·022*	·2	·027	·2	·031	·2	·039
	·2	·022	*·232*	*·026*	*·286*	*·029*	·3	·034
	·3	·028	·3	·027	·3	·029	*·338*	*·033*
	·4	·040	·4	·033	·4	·032	·4	·034
	·5	·058	·5	·044	·5	·039	·5	·038
	·6	·081	·6	·059	·6	·049	·6	·045
.05	·2	·040	·2	·052	·2	·063	·2	·076
	·286	*·038*	·3	·046	·3	·056	·3	·067
	·3	·038	*·372*	*·045*	·4	·052	·4	·062
	·4	·041	·4	·045	*·453*	*·052*	·5	·060
	·5	·050	·5	·048	·5	·052	*·528*	*·060*
	·6	·063	·6	·054	·6	·055	·6	·060
.07	·2	·064	·2	·082	·2	·099	·2	·118
	·3	·056	·3	·072	·3	·089	·3	·107
	·389	*·055*	·4	·067	·4	·082	·4	·098
	·4	·055	*·498*	*·065*	·5	·078	·5	·093
	·5	·057	·5	·065	*·598*	*·077*	·6	·090
	·6	·064	·6	·067	·6	·077	*·686*	*·089*
.10	·2	·105	·2	·131	·2	·159	·2	·184
	·3	·092	·3	·119	·3	·146	·3	·172
	·4	·085	·4	·110	·4	·135	·4	·161
	·5	·082	·5	·104	·5	·128	·5	·154
	·528	*·082*	·6	·101	·6	·123	·6	·147
	·6	·083	*·662*	*·100*	*·772*	*·120*	*·860*	*·141*
.14	·2	·164	·2	·202	·2	·242	·2	·279
	·3	·147	·3	·187	·3	·226	·3	·264
	·4	·136	·4	·175	·4	·214	·4	·252
	·5	·129	·5	·166	·5	·204	·5	·243
	·6	·125	·6	·160	·6	·197	·6	·235
	·686	*·124*	*·828*	*·154*	*·925*	*·186*	*·980*	*·223*

Figures in italics refer to minimum loss of work.

for given clearance and expansion, together with the corresponding amount of compression required. This latter quantity becomes

the smaller, the smaller the clearance space, and as referred to above, when the drying effect of the clearance is taken into account, still less compression means more economy. As will be seen from a glance at the table, the value of m for minimum loss of work also becomes smaller as the ratio of expansion becomes less. "Engines which keep the exhaust port open as long as possible are more economical than those in which there is much compression"—THE ENGINEER for July 12th. 1895. Why this should be so can be understood, at least to some extent. Various curves, represented in Figs. 3 to 8, have been plotted from the results of the above calculations, showing the effects of clearance and compression. These will, it is hoped, tell their own tales without further explanation. In conclusion, it must be confessed that the above investigation is only a rough approximation to the truth, for there are various points of importance that are entirely ignored, such as the drying effect, action of clearance wall as distinguished from the wall of working cylinder, re-evaporation during exhaust, &c.

A. Inokuty (of Japan).

London, August 11th.

V.

Strength of Wheel Teeth.

[*The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. I, No.* 1, 1897.]

In an ordinary form of teeth of a toothed wheel, the existence of two rectangular corners of a tooth means a considerable source of weakness, for from inaccurate form in the teeth or inaccurate fixing of the wheel, the pressure transmitted may happen to act at one corner of a tooth, tending to break it across an oblique section passing through the root at one side. Treating the tooth as a rectangular cantilever fixed at the root and having a uniform thickness equal to the thickness of the tooth at the pitch line, Prof. Unwin gives the following equation for the strength of wheel teeth under the condition above stated:—

$$f_1=\frac{3nP}{t^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

where f_1 is the maximum bending stress occurring in an oblique section AD through the point A at the root, making the angle $BAD=45°$, and nP the pressure acting at the corner B; the total pressure transmitted by the wheel being P, n is a fraction lying between $\frac{1}{2}$ and 1.

When a wheel has an accurate form of teeth and is carefully fitted, the pressure will be distributed along the whole width

of the tooth. The strength of the tooth in this case is given by the equation

$$f_2 = \frac{6nP}{t^2} \cdot \frac{h}{b}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

where f_2 is the maximum bending stress occurring in the section AF along the root. Taking ordinary proportions $h = \cdot 7$ pitch and $b = 2\frac{1}{2}$ pitch, we have

$$\frac{f_1}{f_2} = 1 \cdot 785,$$

that is, the stress due to the pressure acting at one corner is greater by $78\frac{1}{2}$ per cent than that due to the same pressure distributed over the whole edge. In actual wheels, the stress f_1 would be still greater than this, for the thickness along the root is in most cases greater than t, while the thickness at the outer end D of the oblique section is less than t.

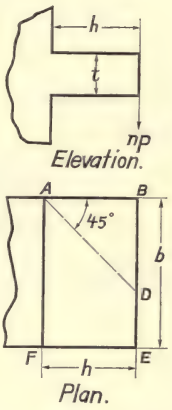


FIG. 1.

Now the fact that in a machine shop of moderate size one can find out without difficulty half a dozen toothed wheels with teeth broken at the corners shews that the existence of such corners is really worse than useless. We see from the following investigation that the strength of wheel teeth is greatly increased by simply cutting away small pieces off the corners.

A triangular portion BCE, Fig. 2, being cut away from the tooth, suppose the pressure to act at the point C tending to break it across an oblique section AD. Using the notations indicated in Fig. 2 the bending moment on the cantilever at the section AD is

$$nP.\overline{CF}$$

i.e., $$nP.\overline{AC}\sin(\theta-\alpha)$$

i.e., $$nP.\overline{AB}.\sec\alpha\sin(\theta-\alpha)$$

or $$nPh\sec\alpha\sin(\theta-\alpha).$$

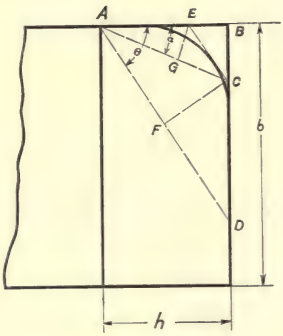


FIG. 2.

The moment of resistance of the section is

$$\tfrac{1}{6}f_1\overline{AD}\,t^2$$

i.e., $$\tfrac{1}{6}f_1 h\sec\theta\times t^2.$$

Equating these to one another, we have

$$nPh\sec\alpha\sin(\theta-\alpha)=\tfrac{1}{6}f_1 ht^2\sec\theta,$$

$$f_1=\frac{6nP}{t^2}\sec\alpha\cos\theta\sin(\theta-\alpha),$$

which will be found to have a maximum value when

$$\theta=\frac{\pi}{4}+\tfrac{1}{2}\alpha$$

and the value of f_1 is then

$$f_1=\frac{3nP}{t^2}\tan\left(\frac{\pi}{4}-\frac{1}{2}\alpha\right). \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

Hence to find the direction AD of the maximum stress f_1, set off CD equal to CA and join AD. Now if CE be parallel to AD, the load nP applied at any point in CE would produce the same stress f_1 in the section AD.

The bending stress f' in the section AC will be greatest when the load nP acts at the point E and as is shown below this greatest tress f' will be less than, equal to, or greater than the stress f_1 in the section AD according as the angle α is less than, equal to, or greater than 30°.

The bending moment at the section AC of the cantilever EAC is

$$n\,P.\,\overline{EG}$$

i.e.,
$$n\,P.\,\overline{AE}.\sin\alpha$$

i.e.,
$$n\,P.\frac{\overline{AC}\sin(\theta-\alpha)}{\sin\theta}.\sin\alpha$$

i.e.,
$$n\,P.\overline{AC}.\frac{\sin\alpha\sin\left(\frac{\pi}{4}-\frac{1}{2}\alpha\right)}{\sin\left(\frac{\pi}{4}+\frac{1}{2}\alpha\right)}.$$

The moment of resistance of the section AC is

$$\tfrac{1}{6}f'.\overline{AC}.t.^2$$

Equating these to one another, we get the stress in the section AC due to the bending action of the load nP applied at the point E,

$$f'=\frac{6\,n\,P}{t^2}\sin\alpha.\frac{\sin\left(\frac{\pi}{4}-\frac{1}{2}\alpha\right)}{\sin\left(\frac{\pi}{4}+\frac{1}{2}\alpha\right)}.$$

From equation (3), the ratio of this stress f' to the stress in the section AD is found to be

$$\frac{f'}{f_1}=\frac{2\sin\alpha\cos\left(\frac{\pi}{4}-\frac{1}{2}\alpha\right)}{\sin\left(\frac{\pi}{4}+\frac{1}{2}\alpha\right)}$$

$$=1+\frac{\sin\left(1\frac{1}{2}\alpha-\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}+\frac{1}{2}\alpha\right)}.$$

This verifies the statement above made. Hence for α not greater than 30°, the following construction should be made. Setting off CD equal to CA, join AD. Draw CE parallel to AD cutting the side AB at E. Cut away the triangular portion BCE. From equation (3) we see that the tooth is thus strengthened in the ratio of $1 : \cot\left(\frac{\pi}{4} - \frac{1}{2}\alpha\right)$. The two corners C and E may then be rounded off by means of a *circular arc touching the three lines AE, EC, and CD, as shown in Fig. 2.

For angle α greater than 30°, the point E must be so chosen as will produce the stress f' in the section AC not to exceed the stress f_1 in the section AD. The bending moment on the cantilever EAC due to the load nP applied at E is

$$n\,P.\,\overline{AE}\sin\alpha.$$

Equating this to the moment of resistance of the section AC, we have

$$n\,P.\,\overline{AE}\sin\alpha = \tfrac{1}{6} f'\,t^2\,h\sec\alpha,$$

from which we get

$$f' = \frac{6\,nP.\,\overline{AE}}{h\,t^2}\cos\alpha\sin\alpha.$$

Equating this to the value of f_1 given by (3), we obtain the greatest safe value of AE,

$$\overline{AE} = \frac{h\tan\left(\frac{\pi}{4} - \frac{1}{2}\alpha\right)}{\sin 2\alpha}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

* The radius of this circular arc is $\dfrac{h\tan\alpha\left(\frac{1}{\sqrt{2}} + \cos\frac{1}{2}\alpha\right)}{\sqrt{2}\cos\left(\frac{\pi}{4} - \frac{1}{2}\alpha\right)}$.

It is worth observing that with an ordinary form of a tooth, if the width b is less than twice the total height h, the tooth is less strong to withstand the pressure distributed over the whole width at the edge than against the same pressure concentrated at one corner of it. For in the first case the stress f_2 induced at the root of the tooth is given by the equation (2) and in the second case the maximum stress f_1 is given by the equation (1), and consequently

$$\frac{f_2}{f_1}=\frac{2h}{b}.$$

Thus if b is less than $2h$ the stress f_2 will be greater than f_1 and the tooth is more liable to break along the root. Hence with the usual proportion of $h=\cdot7\times$pitch, the width b should not be less than $1\cdot4\times$pitch for ordinary form of tooth. If the corners of the tooth are rounded as is here proposed, the width b should not be less than

$$2\,h\cot\left(\frac{\pi}{4}-\frac{1}{2}\alpha\right),$$

i.e.,

$$1\cdot4\,p\cot\left(\frac{\pi}{4}-\frac{1}{2}\alpha\right), \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

which becomes $2\cdot42\,p$ for $\alpha=30°$.

Now making the angle $\alpha=30°$, we obtain $BC=\cdot577\,h$; $BE=\cdot333\,h$; the radius of the circular arc touching AE, EC, and $CD=\cdot789\,h$; again comparing the equations (1) and (2), we obtain the ratio $1:\tan\left(\frac{\pi}{4}-\frac{1}{2}\alpha\right)=1:\cdot577$ in which the stress f_1 is diminished, keeping the pressure and the thickness the same; similarly the ratio $1:\cot\left(\frac{\pi}{4}-\frac{1}{2}\alpha\right)=1:1\cdot732$ in which the pressure can be increased, keeping the two other quantities the same; and lastly if the stress and the pressure remain the same,

the thickness and consequently the pitch p can be reduced in the ratio $1 : \sqrt{\tan\left(\frac{\pi}{4}-\frac{1}{2}\alpha\right)}=1 : 0{\cdot}760$. If we now put $h={\cdot}7p$ and $n=\frac{2}{3}$, the equation (3) becomes

$$p=K\sqrt{P}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

where the constant K may have the values ·036, ·045, and ·057 corresponding to the stress $f_1=6880$, 4400, and 2740 lbs. per sq. inch respectively. The equation (6) gives somewhat smaller pitch than one given by Prof. Unwin, while the corresponding values of the stress just given are considerably lower, showing clearly the advantageous effect of cutting away small pieces off the corners of a tooth.

The following table shows at a glance the results of the above calculation.

Angle α	$\frac{BC}{h}=\tan\alpha$	$\frac{BE}{h}=\tan\alpha\tan\left(\frac{\pi}{4}-\frac{1}{2}\alpha\right)$	Ratio in which f_1 is diminished, P & p remaining the same, $1 : \tan\left(\frac{\pi}{4}-\frac{1}{2}\alpha\right)$.	Ratio in which P can be increased, f_1 & p remaining the same, $1 : \cot\left(\frac{\pi}{4}-\frac{1}{2}\alpha\right)$.	Ratio in which p can be reduced, f_1 & P remaining the same, $1 : \sqrt{\tan\left(\frac{\pi}{4}-\frac{1}{2}\alpha\right)}$.	Minimum value of $\frac{b}{p}$ so that f_2 at root may not exceed f_1 at oblique section, $1{\cdot}4\cot\left(\frac{\pi}{4}-\frac{1}{2}\alpha\right)$.
0°	·0000	·0000	1·0000	1·000	1·000	1·400
5°	·0875	·0802	·9163	1·091	·957	1·527
10°	·1763	·1479	·8391	1·192	·916	1·669
15°	·2679	·2056	·7673	1·303	·876	1·824
20°	·3640	·2549	·7002	1·428	·837	1·999
25°	·4663	·2971	·6371	1·570	·798	2·198
30°	·5774	·3333	·5774	1·732	·760	2·425
35°	·7002	·3645	·5206	1·921	·721	2·689
40°	·8391	·3913	·4663	2·145	·683	3·003
45°	1·000	·4142	·4142	2·414	·644	3·380

VI.

Designing Pressure Blowers and Rotary Pumps.

[*The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. I, No.* 1, 1897.]

The general form of the curves for the outline of the revolving pistons of Roots' pressure blowers and rotary pumps, or in general, of a pair of chamber wheels in gear depends upon two kinematical principles as follows:

1° In order that the velocity ratio of a pair of chamber wheels in gear may be constant, the direction of the common normal at the point of contact of a pair of teeth must pass through the point of contact of the two pitch circles.

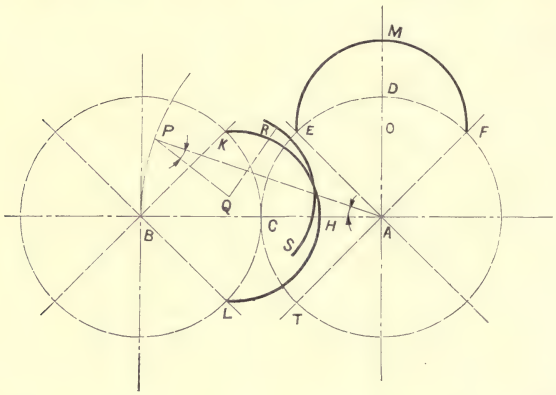
2° In order that the above condition may be satisfied, the form of the tooth curve on the outside of the pitch circle of one wheel must be a certain envelop of the tooth curve on the inside of the pitch circle of the other wheel, and *vice versa*.

If these curves be the roulettes traced out by another curve rolling on the outside of the pitch circle of the first wheel and on the inside of the pitch circle of the second wheel, the tooth curves will be envelops to each other and the condition 1° is satisfied. In ordinary toothed wheels, as is well known, the generating curves for the roulettes are generally circles.

We are quite at liberty to choose any form of tooth curve either for the portion lying on the outside of the pitch circle or for that on the inside, provided that the radius of curvature at the outermost point of a tooth be not greater than that at the corresponding innermost point of the other tooth in gear.

We shall now describe the geometrical construction of the curve for the revolving piston of a Roots' blower. It will not be difficult to draw the curves for other cases by similar geometrical construction.

Let A and B be the centres of two equal pitch circles touching each other at the point C. The height DM of the



tooth outside the pitch circle must be equal to the depth CH inside the pitch circle, and for the case of two lobed curves as in a Roots' blower, these dimensions cannot be made much greater than half the radius of the pitch circle. Having chosen the points H and M at convenient positions, draw through the centres A and B lines making angles of 45° with the centre line AB and cutting the pitch circle at E, F, L, K, &c. These are

then necessarily points on the tooth curves. Now any arbitrary form of curve may be given either to the portion *EMF* lying on the outside of the pitch circle or to the portion *EHT* on the inside. In the figure we have, for simplicity, drawn a circular arc through *EMF*, the centre being at *O*. Then *KHL* is an equal circular arc on the other pitch circle. The curve passing through *EHT* and forming the inside portion of the tooth on the right hand pitch circle must be envelop of the arc *KHL* supposed fixed on the left hand pitch circle *KCL*, while this pitch circle *KCL* itself is rolled round on the outside of the right hand pitch circle as the base. To construct such an envelop, draw a circular arc *BP* with the centre *A* and join *A* with any point *P* on the arc *BP*. Draw *PQ* making the angle *APQ* equal to the angle *BAP*. Set off $PQ=AO$; with *Q* as centre and $QR=OM$ as radius draw the arc *RS*. A series of such circular arcs will form the envelop required. In the accompanying diagrams, a few other forms of chamber wheels are shown.

The volume delivered per revolution by a pair of revolving pistons of a Roots' blower is given by the following formulæ.

Let r=the radius of the two equal pitch circles,

l=the length of the pistons.

Then if the height *DM* of a tooth outside the pitch circle, which is equal to the depth *CH* inside the pitch circle, be made one half the radius of the pitch circle, the capacity per revolution will be found to be

$$C=7\cdot24\,r^2l \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

for the form in which the curve *EMF* outside the pitch circle is a circular arc; and

$$C=2\tfrac{1}{4}\,\pi r^2l=7\cdot07r^2l \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

for the form in which the curve *EMF* is an epicycloid traced out by a circle whose diameter is ¼th of the diameter of the pitch circle.

The following practical rules, proposed by the writer and based on good practice, for the proportion of the various parts of a Roots' blower may prove useful.

The mean velocity of air through the discharge pipe of a fan or pressure blower may be determined by the following empirical formulæ:

$$v = 3\,(d+5) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

for pipes equal to or less than 5″ in diameter; and

$$v = \frac{60\,d}{d+5} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

for pipes equal to or greater than 5″ in diameter. In these formulæ v is the mean velocity of the air in feet per second and d is the diameter of the pipe in inches. The maximum velocity permissible may be taken at about 1¼ times and the minimum at about ¾ times the above value. The following table gives the velocity and the delivery for pipes of different diameters, calculated by the empirical formulæ (3) and (4).

Dia. of pipe in inches	Mean velocity in ft. per sec.	Delivery in cub. ft. per min.	Dia. of pipe in inches	Mean velocity in ft. per sec.	Delivery in cub. ft. per min.
1	18·0	5·76	10	40·0	1310
1½	19·5	14·4	12	42·4	2000
2	21·0	27·5	14	44·2	2830
2½	22·5	46·0	16	45·7	4100
3	24·0	70·7	18	47·0	5220
4	27·0	142	20	48·0	6280
5	30·0	246	24	49·7	9370
6	32·7	385	27	50·6	12100
8	36·9	773	30	51·4	15200

When the pipe is comparatively short it is of course quite possible to employ a much higher velocity without a perceptible loss of pressure.

The pressure of blast obtained by a fan depends on the circumferential velocity at the delivery edge of the fan blade, while the pressure that can be obtained by a Roots' blower does not depend on its speed, so that a Roots' blower of a given capacity per revolution can be adapted to different amounts of delivery per minute by varying its speed of rotation. The following formulæ may, however, be taken to represent the relation between the capacity C of the blower in cub. ft. per revolution and the diameter d of delivery pipe in *feet*, when the blower is working at its normal speed:

$$C = 9{\cdot}26\, d^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

which may be written thus

$$d = 0{\cdot}476 \sqrt[3]{C} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

giving the diameter of the pipe in ft. suitable for a blower of a given capacity in cub. ft. per revolution.

When the blower is driven by belting the diameter of the pulley may be taken equal to $2{\cdot}14\, d$; the width of the pulley may be

$= 0{\cdot}25d + 1\frac{1}{4}''$ for $d \lessgtr 5''$,

$= 0{\cdot}426\, d + 0{\cdot}37''$ for d from $5''$ to $16,''$

$= \frac{1}{36}\, d^2$ for $d \gtrless 16.''$

To find the amount of power required for the delivery of a given quantity of air at a given pressure per unit time, let P be the absolute pressure of air in lbs. per sq. ft. and V the volume in cub. ft. of one lb. of air at pressure P and at a constant temperature. Also let V_a and P_a be the corresponding quantities

under the normal pressure of the atmosphere, so that P_a is the absolute atmospheric pressure itself. Then, as is usual, we have $PV=P_aV_a$. Now the amount of work done in compressing the air from P_a to P is

$$-\int_{V_a}^{V} P\,dV=-\int_{V_a}^{V}\frac{P_aV_a}{V}dV=-P_aV_a\log\frac{V}{V_a}$$

$$=P_aV_a\log\frac{P}{P_a}\text{ ft. lbs.}$$

The work done in the expulsion of the air at pressure P after the compression is PV ft. lbs. which is equal to V_aP_a ft. lbs. But the amount of work done on the pistons of the blower by the pressure of the external air is P_aV_a ft. lbs. Hence the resultant amount of work to be done on the air by the blower is

$$PV\log\frac{P}{P_a}\text{ ft. lbs.} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (A)$$

Now the ratio $P\div P_a$ being only a very little greater than unity,

$$\log\frac{P}{P_a}=2\frac{P-P_a}{P+P_a}\text{ very nearly.}$$

Hence the expression (A) is equal to

$$\frac{2P}{P+P_a}\cdot(P-P_a)\,V\text{ ft. lbs. very nearly.} \quad (A')$$

Putting Q=the volume of air in cub. ft. per min. estimated at the pressure p lbs. per sq. inch above the atmosphere, the theoretical horse power required is

$$H.P.=\frac{2(p+p_a)}{p+2p_a}\cdot\frac{144\,pQ}{33000}, \quad \ldots\ldots\ldots\ldots \quad (7)$$

where p_a=14·7 lbs. per sq. inch.

At the mean temperature of 60°*F.*, one cub. ft. of air weighs 0·076097 lbs. under the atmospheric pressure and, therefore, under a pressure of p lbs. per sq. inch above the atmosphere, it will weigh

$$w = \frac{p+p_a}{p_a} \times 0{\cdot}076097 \text{ lbs.} \quad \ldots\ldots\ldots\ldots\ldots \quad (8)$$

Supposing that the pressure of the air in the pipe is not perceptibly reduced till it reaches the discharge end of the pipe, the velocity of efflux of the air from the nozzle is

$$v = \sqrt{2g \times \frac{144p}{w}} \text{ ft. per second.} \quad \ldots\ldots\ldots \quad (9)$$

Dividing the discharge Q by this velocity v, we obtain the effective area of the nozzle.

As an example let it be required to find the principal dimensions of a Roots' blower for supplying the blast to a cupola capable of melting 4 tons of cast iron per hour.

The volume of air required for this cupola will be about 2000 cub. ft. per min., so that from the table given above, the diameter of the discharge air main will be found to be 12.″ From formula (5), we obtain the capacity of the blower,

$$C = 9{\cdot}26 \text{ cub. ft. per revolution.}$$

The number of revolutions of the blower is then $2000 \div 9{\cdot}26 = 216$ per min. The length l of the revolving piston may be from $4\frac{1}{2}$ to 6 times the radius of the pitch circle; we will take it at 5 times. Using the formula (1), we have

$$r = \sqrt{\frac{9{\cdot}26}{7{\cdot}24 \times 5}}$$

$$= {\cdot}635 \text{ ft.} = 7{\cdot}62.''$$

The diameter of the piston chamber $=3r=22{\cdot}85''$.
The length of revolving piston $=5r=1$ ft. $11''$.
The diameter of driving pulley $=2{\cdot}14\times12=25\frac{1}{2}''$.
The width of driving pulley $=0{\cdot}426\times12+{\cdot}37=5\frac{1}{2}''$.

The maximum pressure of the blast required by a cupola will be about 0·78 lb. per sq. inch above the atmosphere. The theoretical power required is obtained from the formula (7).

$$\text{Theoretical}\quad H.P.=\frac{2({\cdot}78+14{\cdot}7)}{{\cdot}78+2\times14{\cdot}7}\times\frac{144\times{\cdot}78\times2000}{33000}$$

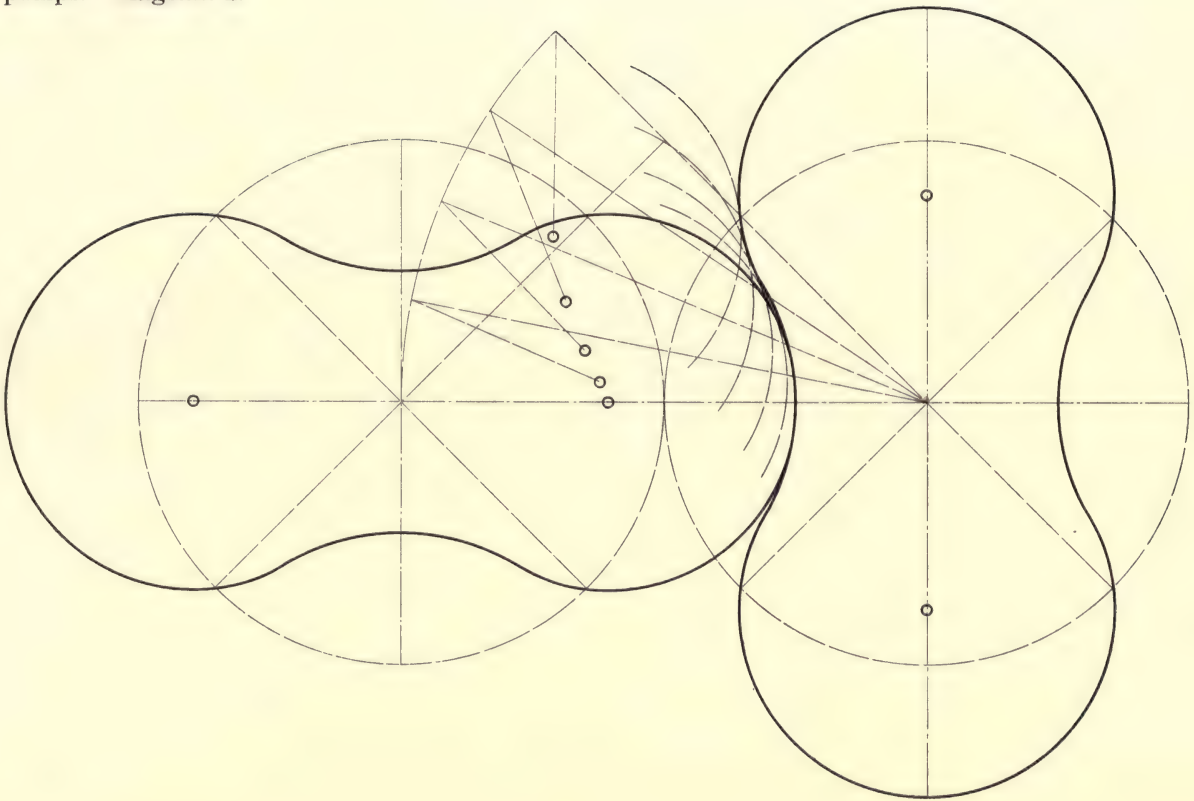
$$=1{\cdot}026\times6{\cdot}37=6{\cdot}53.$$

If we assume an efficiency of 75 per cent., the actual horse power required would be 8·7. The weight of air under a pressure of ·78 lb. per sq. inch at 62° *F.* is by (8) $={\cdot}08009$ lb. per cub. ft. The velocity of the blast at the nozzle is then

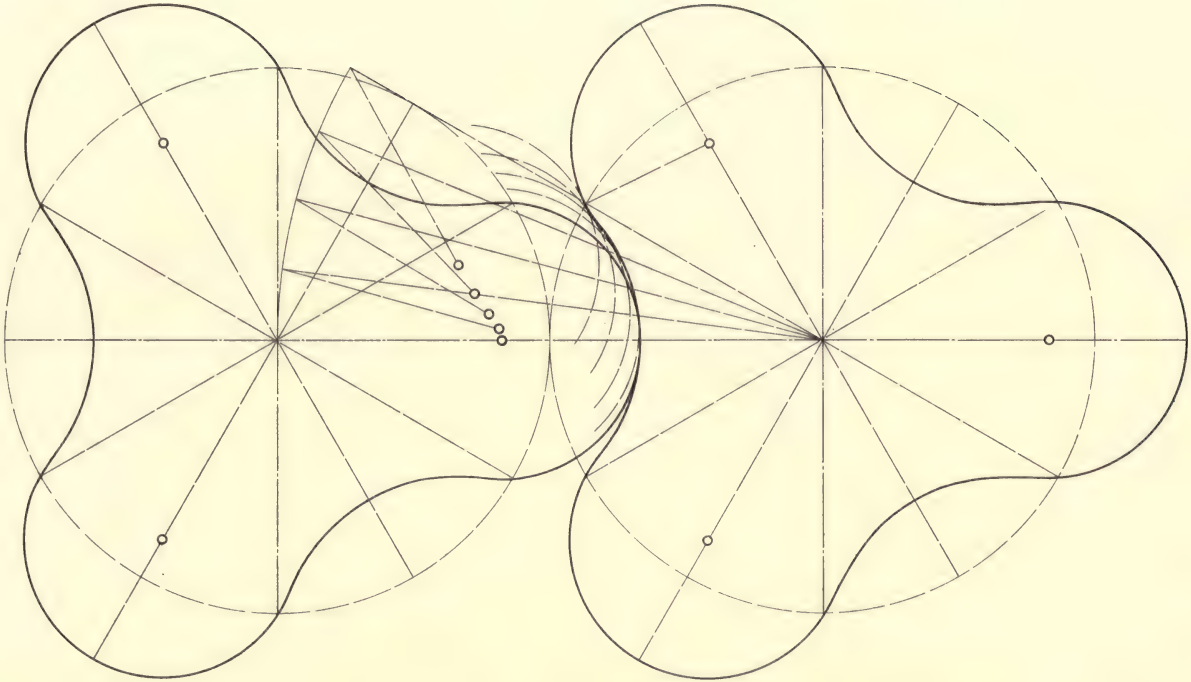
$$v=\sqrt{2g\times\frac{144\times{\cdot}78}{{\cdot}08009}}=300 \text{ ft. per second.}$$

The effective area of the nozzle is then found to be 16 sq. inches, which is the area of a circle $4\frac{1}{2}''$ in diameter.

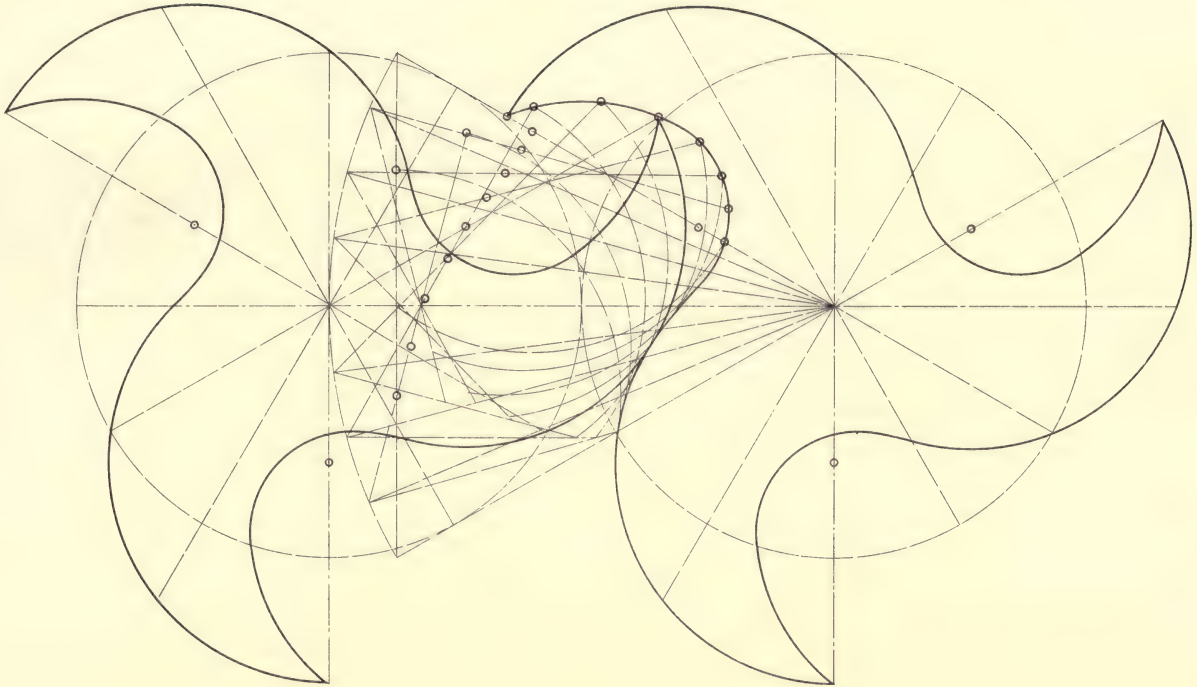
Designing pressure blowers and rotary pumps. Diagram 1.



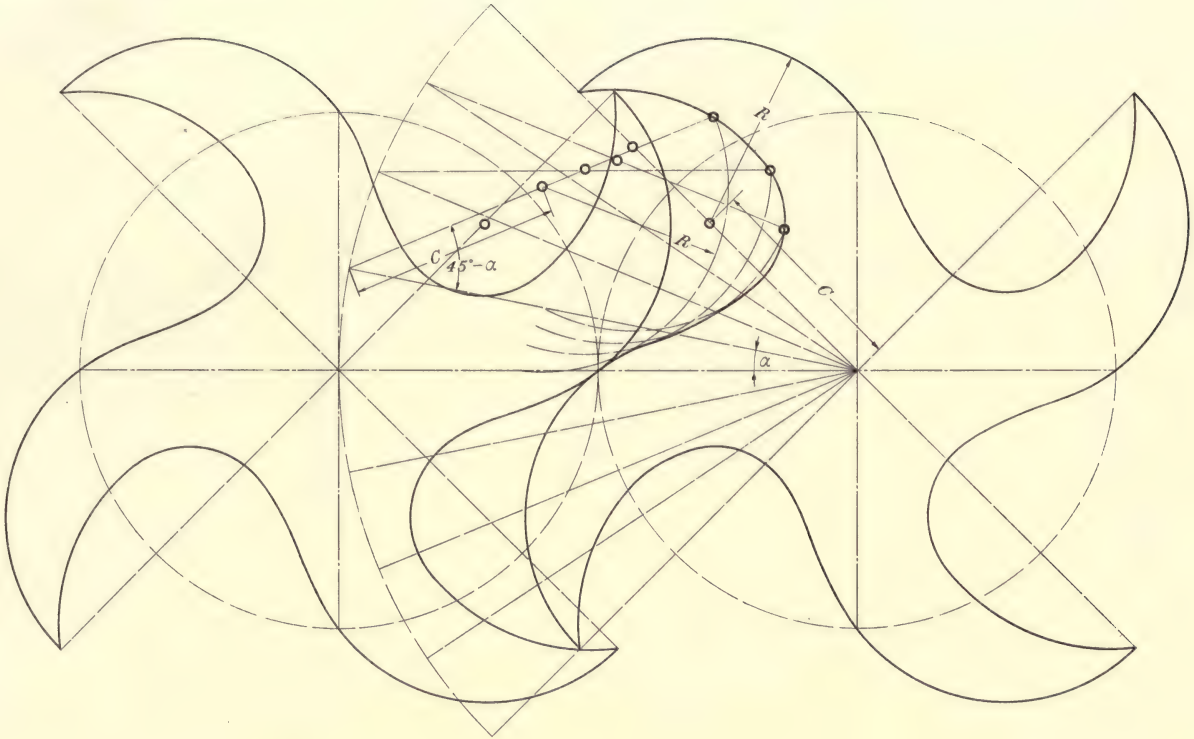
Designing pressure blowers and rotary pumps. Diagram 2.



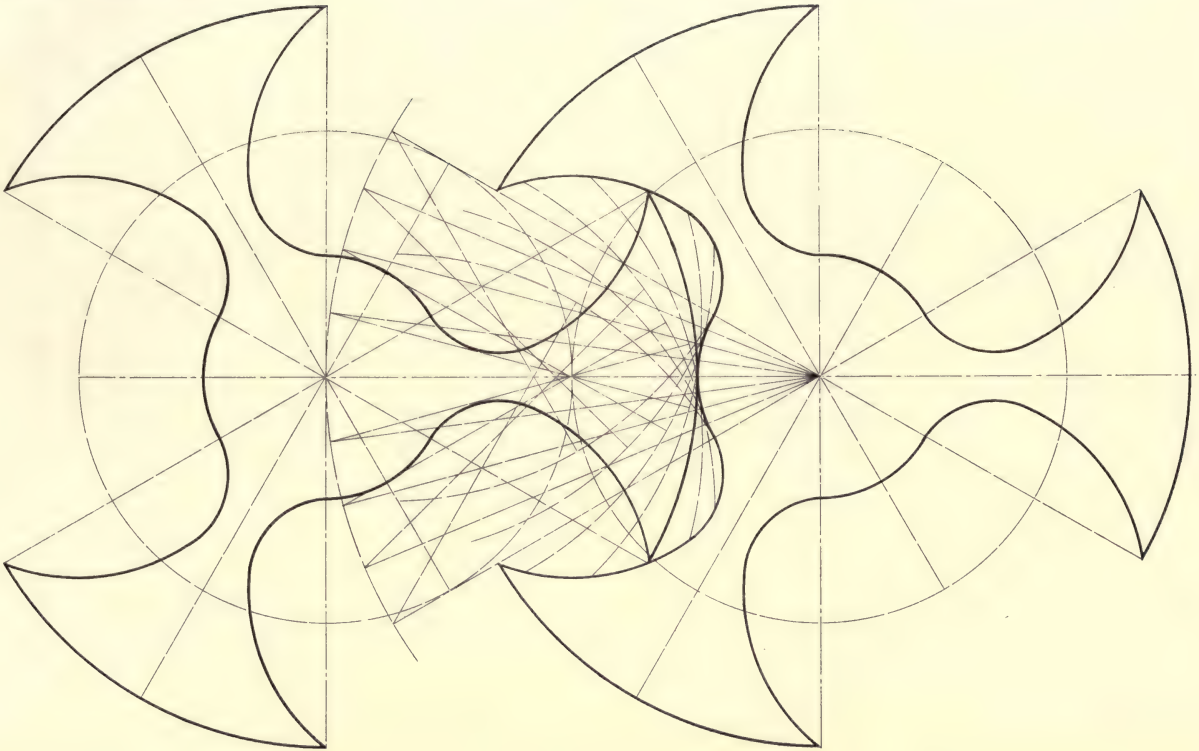
Designing pressure blowers and rotary pumps. Diagram 3.



Designing pressure blowers and rotary pumps. Diagram 4.



Designing pressure blowers and rotary pumps. Diagram 5.



VII.

Effect of Inertia of Moving Parts of an Engine on Piston and Crank Pin Effort.

[*Transaction of Japanese Institution of Naval Architects, No.* 1, *December,* 1897.]

In high speed engines, the inertia of the massive parts directly connected with the piston and reciprocating with it, and also of the connecting rod and the unbalanced part of the crank, greatly alters the piston and crank pin effort. The following is an exact method of determining the various forces arising from the acceleration of moving parts.

The mechanics of the connecting rod will be considered first and then the result of it will be applied to the calculations of inertia effect of other moving parts. Referring to Fig. 1, let $AB=R$ be the crank radius; $BC=L$ the connecting rod length; P any point in the centre line BC of the connecting rod, distant l from the crank pin B. Let V be the velocity of the crank pin B; $\theta=$ the angle CAB which the crank makes with the line of stroke AC; $\alpha=$ the angle BCA; and $t=$ the time measured from the beginning of the stroke.

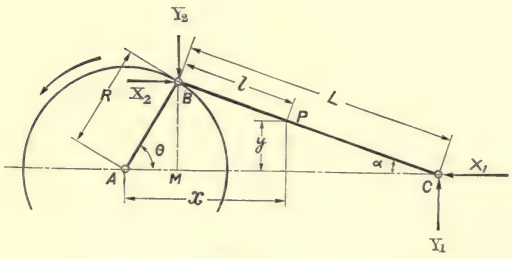


FIG. 1.

The angular velocity of the crank, which we assume to be uniform, is $\omega = V \div R = \frac{d\theta}{dt}$. Taking A for the origin and AC for the axis of abscissa, the coordinates of the point P are

$$\left.\begin{aligned} x &= R\cos\theta + l\cos\alpha \\ y &= R\sin\theta - l\sin\alpha \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

The two angles θ and α have the relation

$$\sin\alpha = \frac{1}{n}\sin\theta, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

where n is put for the ratio $L \div R$ for simplicity. From this, we have

$$\frac{d\alpha}{dt} = \frac{\cos\theta}{(n^2 - \sin^2\theta)^{\frac{1}{2}}}\,\frac{d\theta}{dt}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

Differentiating again and observing that $\frac{d^2\theta}{dt^2} = 0$, we obtain

$$\frac{d^2\alpha}{dt^2} = -\frac{(n^2-1)\sin\theta}{(n^2-\sin^2\theta)^{\frac{3}{2}}}\left(\frac{d\theta}{dt}\right)^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Now differentiating the first of the equations (1), we have

$$\frac{dx}{dt} = -R\frac{d\theta}{dt}\sin\theta - l\frac{d\alpha}{dt}\sin\alpha.$$

Differentiating again, we obtain

$$\frac{d^2x}{dt^2} = -R\left(\frac{d\theta}{dt}\right)^2 \cos\theta - l\left(\frac{da}{dt}\right)^2 \cos a - l\frac{d^2a}{dt^2}\sin a. \quad \ldots\ldots\ldots \quad (5)$$

Substituting the expressions given by (2), (3), and (4) and simplifying, we obtain the equation

$$\frac{d^2x}{dt^2} = -\left(\frac{d\theta}{dt}\right)^2\left\{R\cos\theta + \frac{l}{n}\cdot\frac{n^2\cos 2\theta + \sin^4\theta}{(n^2-\sin^2\theta)^{\frac{3}{2}}}\right\}. \quad \ldots\ldots\ldots \quad (5')$$

This is an expression for the resolved part of the acceleration parallel to the line of stroke, of any point in the centre line of the connecting rod; by making l equal to L we obtain the translational acceleration of the piston and other parts directly connected with it; and by making l equal to zero we obtain the component acceleration of the crank pin parallel to the line of stroke. In particular, putting $l=\bar{l}=$ distance of the centre of gravity of the connecting rod from the crank pin end, we obtain the component acceleration of that point parallel to the line of stroke, which may be written thus:—

$$\frac{d^2\bar{x}}{dt^2} = -\left(\frac{d\theta}{dt}\right)^2\left\{R\cos\theta + \frac{\bar{l}}{n}\cdot\frac{n^2\cos 2\theta + \sin^4\theta}{(n^2-\sin^2\theta)^{\frac{3}{2}}}\right\}. \quad \ldots\ldots\ldots \quad (6)$$

Differentiating the second of the equations (1), we obtain

$$\frac{dy}{dt} = R\frac{d\theta}{dt}\cos\theta - l\frac{da}{dt}\cos a.$$

Differentiating again and remembering that $\frac{d^2\theta}{dt^2}=0$, we obtain

$$\frac{d^2y}{dt^2} = -R\left(\frac{d\theta}{dt}\right)^2\sin\theta + l\left(\frac{da}{dt}\right)^2\sin a - l\frac{d^2a}{dt^2}\cos a. \quad \ldots\ldots\ldots \quad (7)$$

Making substitutions from (2), (3), and (4) we obtain

$$\frac{d^2y}{dt^2} = -\left(\frac{d\theta}{dt}\right)^2\left(R-\frac{l}{n}\right)\sin\theta, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7')$$

which is the component acceleration at right angles to the line of stroke, of any point in the centre line of the connecting rod. This result may be written down at once from Fig. 1. Making $l=\bar{l}$, we obtain

$$\frac{d^2\bar{y}}{dt^2}=-\left(\frac{d\theta}{dt}\right)^2\left(R-\frac{\bar{l}}{n}\right)\sin\theta. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

Now leaving out of consideration the force of gravity on the moving masses, there are only two other external forces acting on the connecting rod, one at the crosshead pin and the other at the crank pin. These may be decomposed parallel and perpendicular to the line of stroke. Let these be represented by X_1, X_2, Y_1, and Y_2 as indicated in Fig. 1. The dynamical equations of motion for the connecting rod are then as follows:—

$$M\frac{d^2\bar{x}}{dt^2}=X_2-X_1, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

$$M\frac{d^2\bar{y}}{dt^2}=Y_1-Y_2, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

and

$$I_0\frac{d^2\alpha}{dt^2}=(L-\bar{l})\sin\alpha\, X_1+\bar{l}\sin\alpha\, X_2 \\ -(L-\bar{l})\cos\alpha\, Y_1-\bar{l}\cos\alpha\, Y_2. \quad \ldots\ldots \quad (11)$$

Since we are considering the forces arising from acceleration only and the external force causing this acceleration is supposed to be supplied by a part of the pressure on the piston through the instrumentality of the crosshead pin, it is evident that there can be no resistance in the direction of motion of the crank pin, acting on the connecting rod at the crank pin. We have, therefore, equation of condition

$$X_2\sin\theta+Y_2\cos\theta=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

In the equations above, M is the mass of the connecting rod and I_0 its moment of inertia about an axis passing through the centre of gravity and perpendicular to the plane of motion.

If, in any particular engine, the pressure on piston becomes insufficient to produce the force X_1 at the crosshead pin, the defficiency will be supplied by the inertia of the crank shaft and other rotating part, acting through the crank arm and finally at the crank pin. In this case the equation (12) does not hold; in fact the piston effort and consequently the crank pin effort will here be negative.

Again if the pressure on the piston become nothing or negative as in most gas engines and oil engines, it is the inertia of the flywheel and crank shaft that causes the acceleration of the connecting rod. It is evident that in this case there can be no resistance in the direction of motion of the crosshead pin, acting on the connecting rod at the crosshead pin. We have, therefore, an equation of condition

$$X_1=0. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (12')$$

From the four equations, namely, (9), (10), (11), and (12), we can determine the four component forces and then knowing the pressure on piston, a complete investigation relating to the forces and speed can be made. Take the equation (11) first. It can be written thus:—

$$I_0\frac{d^2a}{dt^2}=L\sin a\, X_1-L\cos a\, Y_1+\bar{l}\sin a(X_2-X_1)+\bar{l}\cos a\,(Y_1-Y_2).$$

Substituting the values of X_2-X_1 and Y_1-Y_2 given by the equations (9) and (10), we obtain

$$I_0\frac{d^2a}{dt^2}-\bar{l}M\left(\sin\alpha\frac{d^2\bar{x}}{dt^2}+\cos\alpha\frac{d^2\bar{y}}{dt^2}\right)=L(\sin\alpha X_1-\cos\alpha Y_1).$$

Making substitutions from equations (5) and (7), we obtain after reduction

$$I_2\frac{d^2a}{dt^2}+MR\omega^2\bar{l}\sin(\theta+\alpha)=L(\sin\alpha X_1-\cos\alpha Y_1), \quad \ldots\ldots\ldots \quad (13)$$

where $I_2=I_0+\bar{l}^2M=$ the moment of inertia of the connecting rod about the axis of the crank pin as the axis. From the equation of condition (12), we have

$$Y_2=-X_2\tan\theta.$$

Putting this value of Y_2 in the equation (10), we have

$$M\frac{d^2\bar{y}}{dt^2}=Y_1+X_2\tan\theta.$$

Eliminating X_2 between this and the equation (9), we have

$$M\left(\cos\theta\frac{d^2\bar{y}}{dt^2}-\sin\theta\frac{d^2\bar{x}}{dt^2}\right)=X_1\sin\theta+Y_1\cos\theta.$$

Substituting the expressions given by (5′) and (7′), we obtain after reduction

$$MR\omega^2K\bar{l}\sin\theta=L(X_1\sin\theta+Y_1\cos\theta), \quad \ldots\ldots\ldots\ldots \quad (14)$$

where for simplicity we put

$$K=\cos\theta+\frac{n^2\cos 2\theta+\sin^4\theta}{(n^2-\sin^2\theta)^{\frac{3}{2}}}.$$

Eliminating Y_1 between (13) and (14), we obtain

$$LX_1\sin(\theta+\alpha)$$
$$=I_2\frac{d^2a}{dt^2}\cos\theta+\bar{l}MR\omega^2\{K\sin\theta\cos\alpha+\cos\theta\sin(\theta+\alpha)\}. \quad \ldots \quad (15)$$

This determines the value of X_1. Eliminating X_1 between (13) and (14), we obtain

$$L Y_1 \sin(\theta+\alpha)$$
$$= I_2 \frac{d^2\alpha}{dt^2} \sin\theta + \bar{l} M R\omega^2 \Big\{ K \sin\theta \sin\alpha + \sin\theta \sin(\theta+\alpha) \Big\}. \quad \ldots \quad (16)$$

This determines the value of Y_1. The equation (11) can be written in the form

$$I_0 \frac{d^2\alpha}{dt^2} = (L-\bar{l}) \sin\alpha (X_1 - X_2) - (L-\bar{l}) \cos\alpha (Y_1 - Y_2)$$
$$+ L \sin\alpha X_2 - L \cos\alpha Y_2.$$

Substituting the value given by (9) and (10), we have

$$I_0 \frac{d^2\alpha}{dt^2} + (L-\bar{l}) M \left(\sin\alpha \frac{d^2\bar{x}}{dt^2} + \cos\alpha \frac{d^2\bar{y}}{dt^2} \right) = L(\sin\alpha X_2 - \cos\alpha Y_2).$$

Again substituting the expressions given by (5) and (7), we obtain

$$\left\{ I_0 - (L-\bar{l})\bar{l} M \right\} \frac{d^2\alpha}{dt^2} - (L-\bar{l}) M R\omega^2 \sin(\theta+\alpha)$$
$$= L(\sin\alpha X_2 - \cos\alpha Y_2).$$

But the equation of condition (12) gives

$$Y_2 = -X_2 \tan\theta.$$

Hence we obtain

$$L X_2 \sin(\theta+\alpha)$$
$$= \left\{ I_0 - (L-\bar{l}) M \bar{l} \right\} \frac{d^2\alpha}{dt^2} \cos\theta - (L-\bar{l}) M R\omega^2 \cos\theta \sin(\theta+\alpha). \quad (17)$$

This determines the value of X_2. These equations are too complex for practical application.

If we assume as an approximate truth that the force arising from the translational acceleration of the connecting rod parallel

to the line of stroke acts at its centre of gravity, then the equation (9) at once enables us to determine that force; for using the value of the acceleration given by (6), we have the translational resistance of the connecting rod equal to

$$\frac{1}{g}W_c\left(\frac{d\theta}{dt}\right)^2\left\{R\cos\theta+\frac{\bar{l}}{n}\,\frac{n^2\cos 2\theta+\sin^4\theta}{(n^2-\sin^2\theta)^{\frac{3}{2}}}\right\},$$

where W_c is the total weight of the connecting rod. But since $nR=L$, if we put $.\bar{l}=\bar{l}\div L$, the above can be written as follows:—

$$-\frac{1}{g}(1-.\bar{l})W_cR\omega^2\cos\theta-\frac{1}{g}\,.\bar{l}\,W_cR\omega^2\left\{\cos\theta+\frac{n^2\cos 2\theta+\sin^4\theta}{(n^2-\sin^2\theta)^{\frac{3}{2}}}\right\}.$$

The last expression within the twisted brackets is the same factor as in the expression for the acceleration of crosshead and piston. Hence we conclude that so far as the translational inertia of the connecting rod parallel to the line of stroke is concerned, the effect is just the same as if its entire mass were concentrated at the centres of the crosshead pin and the crank pin in the inverse ratio of the segments into which the centre of gravity divides the length between the two centres.

Now let W_p= the weight of the piston, piston rod, and crosshead; W_k= the weight of the unbalanced part of the crank webs, eccentric discs, &c. Then the total resistance due to translational acceleration of the whole moving parts may be separated into two parts. The first part, which is due to the simple harmonic motion in the line of stroke, is

$$\frac{1}{g}\left\{.xW_k+(1-.\bar{l})W_c+.\bar{l}W_c+W_p\right\}R\omega^2\cos\theta,$$

i. e.

$$\frac{1}{g}(.xW_k+W_c+W_p)R\omega^2\cos\theta, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (A)$$

and the second part, which arises from the angularity of the connecting rod, is

$$\frac{1}{g}(\cdot\bar{l}W_c + W_p)R\omega^2 \frac{n^2\cos 2\theta + \sin^4\theta}{(n^2-\sin^2\theta)^{\frac{3}{2}}}. \quad \ldots\ldots\ldots\ldots\ldots \quad (B)$$

In these exprssions $.r$ is the distance of the centre of gravity of W_k from the axis of the crank shaft, expressed in fractions of crank length. Putting $x = R(1-\cos\theta)$ for the distance through which the projection of the crank pin on the line of stroke moves from the beginning of stroke, the expression (A) takes the form

$$\frac{1}{g}(.rW_k + W_c + W_p)\omega^2(R-x). \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (A')$$

Hence this part of the resistance can be represented by a straight line. In Fig. 2, let AOB be the diameter of the crank pin circle on the line of stroke; OP a position of crank; PM a perpendicular from P on to the line of stroke. Take $CB = AD =$ the expression (A') with x equal to zero and $2R$, respectively, at right angles to AOB. Join CD. The ordinate of this straight line, then, represents the resistance (A). Now for the second part (B), we first transform the term containing θ, thus:—

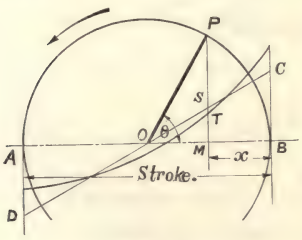


FIG. 2.

$$f(\theta) = \frac{n^2\cos 2\theta + \sin^4\theta}{(n^2-\sin^2\theta)^{\frac{3}{2}}}$$

$$= \frac{n^2(\cos^2\theta - \sin^2\theta) + \sin^2\theta(1-\cos^2\theta)}{(n^2-\sin^2\theta)^{\frac{3}{2}}}$$

$$= \frac{\cos^2\theta(n^2-\sin^2\theta) - (n^2-1)\sin^2\theta}{(n^2-\sin^2\theta)^{\frac{3}{2}}}$$

$$= \frac{\cos^2\theta - (n^2-1)\tan^2\alpha}{n\cos\alpha}$$

$$= \frac{1}{nR}\left\{\frac{R\cos^2\theta - \frac{(\sqrt{L^2-R^2}\tan\alpha)^2}{R}}{\cos\alpha}\right\}.$$

Now in Fig. 3, let OP be the crank, PC the connecting rod, and OC the line of stroke. Draw OD at right angles to OC and meeting the crank pin circle in D. From D set off DE equal to the connecting rod length, cutting the line of the stroke in E and draw EF parallel to CP and meeting OD in F. Take OG equal to OF and draw HG parallel to AF. Then we have

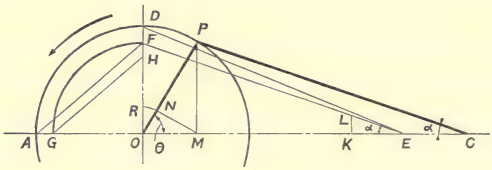


FIG. 3.

$$OF = OE\tan\alpha$$

$$= \sqrt{DE^2 - OD^2}\tan\alpha$$

$$= \sqrt{L^2-R^2}\tan\alpha.$$

But by construction, we have

$$\frac{OH}{OG} = \frac{OF}{OA},$$

$$\therefore \quad OH = \frac{OF\,.\,OG}{OA} = \frac{OF^2}{R}$$

$$= \frac{(\sqrt{L^2-R^2}\tan\alpha)^2}{R}.$$

Now draw PM perpendicular to OC and MN perpendicular to OP. Then we have

$$ON = OM\cos\theta$$
$$= (OP\cos\theta)\cos\theta$$
$$= R\cos^2\theta.$$

Hence we have $$f(\theta) = \frac{1}{nR}\cdot\frac{ON - OH}{\cos\alpha}.$$

Take OR equal to ON. Set off EK equal to the length RH and draw KL at right angles to EKO, meeting EF in L. Then we have

$$EL = \frac{ON - OH}{\cos\alpha}.$$

The expression (B), therefore, becomes

$$\frac{1}{ng}(.\bar{l}W_c + W_p)\omega^2\overline{EL}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (B')$$

If, in Fig. 2, the accelerations CB and AD at the dead centres were drawn to the scale

$$CB = AD = \frac{n(.rW_k + W_c + W_p)}{.\bar{l}W_c + W_p}R,$$

then in order to obtain the curve of resistance due to translational acceleration, we have simply to set off ST in Fig. 2 equal to EL in Fig. 3. A number of such points as T can be found to complete the curve. The effective piston effort diagram and the corresponding crank pin effort diagram can now be drawn without difficulty.

In modern high speed single acting steam engines and in gas and oil engines, the effect of inertia of moving parts forms an important matter, and some of the equations above given may be of use in checking the result of ordinary calculation or graphical determination of the forces due to acceleration.

VIII.

Preliminary Experiment on the Strength of Bolt and Nut.

[*The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. I, No.* 2, 1898.]

An ordinary triangular threaded screw bolt working in a nut whose thickness is from $\frac{3}{4}$ to 1 diameter of the bolt, subjected to direct tensile load, always breaks by tearing in the screwed part especially near the shank. The bolt breaks by tension, because as is wellknown the nut of an ordinary proportion is far stronger than the bolt under tension. It breaks especially near the shank, that is, at the base of the screwed part, because there is a sudden change of the distribution of tensile stress in passing from the screwed part to the shank.

Now supposing that the shearing strength of the screw thread in the direction of the axis of the bolt is equal to the tensile strength of the bolt and that the cylindrical section of the thread at its root is subjected to pure shearing action, the least thickness of the nut which ensures sufficient strength to resist both the tensile and shearing stress would be $\frac{1}{4}$th of the diameter at the bottom of the screw thread. Again supposing that the total tensile load acts uniformly over the surface of the

thread in the direction of the axis of the bolt and that the thread resists this load by behaving as a cantilever of which the length is the depth of the thread, the width is the circumference at the bottom of the thread, and the height at the fixed end is the thickness of the thread at the root, the least thickness of the nut which ensures sufficient strength to resist both the tensile and bending stress would be given by

$$h=\frac{3}{8}\frac{f_t}{f_b}\frac{d_1(d-d_1)}{p}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

where f_t and f_b are the tensile and bending stress induced, respectively; d the diameter of the bolt; d_1 that at the bottom of the thread; and p the pitch of the screw.

In the following table are given the thicknesses of nuts calculated under these hypotheses for several diameters of bolts, the tensile, shearing, and bending stresses all being supposed equal to one another. The screw threads are according to Whitworth's standard sizes.

[TABLE.

Table I.

Diameter of bolt in inches. d	Number of threads per inch.	Least thickness of nut to resist		
		shearing, in inches. $h=\frac{1}{4}d_1$	bending, by (1)	
			in inches. h	in fractions of diameter. $\frac{h}{d}$
$\frac{1}{2}$	12	·0983	·187	·378
$\frac{5}{8}$	11	·127	·245	·392
$\frac{3}{4}$	10	·156	·299	·398
$\frac{7}{8}$	9	·183	·351	·401
1	8	·210	·403	·403
$1\frac{1}{8}$	7	·236	·453	·402
$1\frac{1}{4}$	7	·267	·513	·410
$1\frac{3}{8}$	6	·290	·559	·407
$1\frac{1}{2}$	6	·322	·619	·413
$1\frac{5}{8}$	5	·342	·656	·404
$1\frac{3}{4}$	5	·375	·709	·405
$1\frac{7}{8}$	$4\frac{1}{2}$	·398	·765	·408
2	$4\frac{1}{2}$	·479	·792	·396

It would be very complicated and difficult, if not impossible, to determine by calculation the greatest real stress existing near the root of a screw thread when the bolt is in tension.

Whatever may be the nature and distribution of this stress, it will be possible to determine by actual testing the least thickness of a nut for which the bolt has just enough strength to resist this as well as direct tensile stress. The object of the present experiment is to find out such a proportion of a nut to the diameter of the bolt.

Nine wrought iron bolts, two 1″ in diameter, two $\frac{3}{4}$″ in diameter, and five $\frac{5}{8}$″ in diameter all 4″ long, with ordinary triangular threads cut in a lathe, were prepared. The heights of the bolt heads were all made equal to the diameters and the widths of the nuts were in accordance with the usual proportions, but the thicknesses of the latter were varied. A special grip, Fig. 2, for holding the bolt without any lateral constraint was employed. The tensile load was applied between the nut and the bolt head by Riehlé Brothers' hydraulic testing machine capable of exerting a maximum force of 42000 lbs. The following table gives the results of this testing.

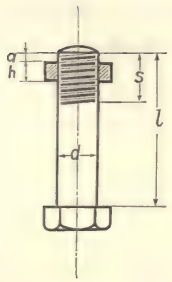


Fig. 1.

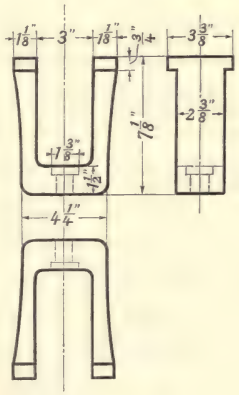


Fig. 2.

Table II. Test on Wrought Iron Bolts and Nuts.

No.	Diameter of bolt d in inches.	Length of bolt l in inches.	Length of screwed part in inches, s.	Part projecting outside the nut, a in inches.	Thickness of nut h in inches.	Thickness of nut in fractions of diameter of bolt, $\frac{h}{d}$.	Total breaking load in lbs.	Total breaking load ÷ area at bottom of thread.	Remarks.
1	1	4	$1\frac{1}{4}$	·25	·563	·563	32380	58,400	Broken by tension at the base of screwed part; fibres strained by tension all along the bottom of thread visible.
2	1	4	$1\frac{1}{4}$	·25	·383	**·383**	25280	45,600	Screw thread stripped and the nut burst out.
3	$\frac{3}{4}$	4	$1\frac{1}{4}$	·20	·383	·511	17200	56,600	Broken by tension underneath the nut. A crystalline flaw at the fractured section.
4	$\frac{3}{4}$	4	$1\frac{1}{4}$	·20	·316	**·421**	16700	55,000	Screw thread stripped.
5	$\frac{5}{8}$	4	$1\frac{1}{2}$	·19	·452	·723	12730	62,800	Broken by tension at the base of screwed part.
6	$\frac{5}{8}$	4	$1\frac{1}{2}$	·19	·383	·613	11120	54,900	Broken by tension close to the nut; a considerable flaw at the fractured section.
7	$\frac{5}{8}$	4	$1\frac{1}{2}$	·19	·315	·504	12000	59,200	Broken by tension at the base of screwed part.
8	$\frac{5}{8}$	4	$1\frac{1}{2}$	·19	·265	**·424**	12040	59,400	Screw thread stripped.
9	$\frac{5}{8}$	4	$1\frac{1}{2}$	·19	·205	·328	9900	48,800	Screw thread stripped.

The figures in heavy type do not differ much from those calculated by (1) and given in the last column of Table I. Further experiments are being made by the writer.

IX.

Strength of Long Tubes under External Pressure.

[*The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. I, No.* 2, 1898.]

A thin cylindrical tube with its ends free and under a uniform external pressure is in a condition similar to that of a long column subjected to an axial load. Were a tube perfectly circular in form, of perfectly uniform thickness, made of perfectly homogeneous material, and subjected to perfectly uniform external pressure, the tube would maintain its form, the stress produced being a simple compressive stress distributed uniformly in a circumferential direction. But when the pressure exceeds a certain limit, the tube is in unstable equilibrium and any slight deviation from perfect accuracy in the above conditions will cause the tube to yield by collapsing. Collapsing in one part is accompanied by bulging in the neighbouring part, so that if the ends of the tube were free as in smoke tubes of steam boiler without beading over, the tube will be divided up into four segments.

A formula corresponding to Euler's formula for long columns has been obtained by Lévy for the collapse of a thin cylindrical tube of an unlimited length,

$$p=\frac{E}{4}\left(\frac{t}{r}\right)^3, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

where t is the thickness of the plate forming the tube, r the mean radius of the tube, E the modulus of direct elasticity and p the external fluid pressure.

Now when the thickness of the tube is large as compared with the radius, the stress will be nearly uniform, because there will then be little tendency to collapse. The formula for this case is very nearly

$$p=f\left(\frac{t}{r}\right). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

An empirical formula may be constructed, approximately satisfying the above equations in the two extreme cases of a very thin tube and a considerably thick tube, thus:

$$p=\frac{f\left(\frac{t}{r}\right)}{1+a\left(\frac{r}{t}\right)^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

This may be called a Gordon formula for a long tube under external pressure, in which the value of the constant a has to be determined by experiments. If we, however, put $a=\frac{4f}{E}$, the equation becomes equivalent to Lévy's formula for an extremely thin tube. This value, not depending upon direct experiments on tubes, may be called an hypothetical value of the constant a. The corresponding hypothetical value of the constant a' in Gordon-Rankine formula

$$p=\frac{f}{1+4a'\left(\frac{l}{k}\right)^2}$$

for a long column rounded at both ends, is given by

$$4a' = \frac{f}{\pi^2 E}.$$

Now it is natural to suppose that the actual value of a is similar to that for a long column. Hence we obtain

$$a = 16\pi^2 a'. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

The following table gives the values of the constant a calculated by the equation (4).

	Breaking stress f in lbs. per sq. inch.	Values of $\frac{1}{a'}$ for columns.	Values of $\frac{1}{a}$ for tubes.
Wrought iron	36,000	36,000	228
Mild steel	42,000	36,000	228
Cast iron	80,000	6,400	40·5
Timber	7,200	3,000	19·0

It is well to notice that Lévy's theoretical formula ceases to be applicable when the ratio of the diameter of a tube to its thickness is less than the limit,

$$\frac{\text{diameter}}{\text{thickness}} = \sqrt{\frac{E}{f}}.$$

For wrought iron, taking E=29,000,000 and f=36,000 lbs. per sq. inch, we obtain the limiting ratio equal to 28·4.

As an example let us find the probable factor of safety for a wrought iron tube 3¾″ in external diameter, 8 *B. W. G.* (=0·165″) thick and 7ft. 6″ long, used as smoke tube of a marine boiler with steam working at 160 lbs. per sq. inch. Here the mean radius of the tube is 1·793″; using formula (3), we obtain the collapsing pressure equal to 2220 lbs. per sq. inch. Dividing this by the working pressure of steam, we obtain 14 very nearly as the probable factor of safety.

X.

Interpolation.

[*Hitherto unpublished.*]

Theory of Difference. Consider a series of numbers proceeding according to a certain law. By subtracting each term from that which follows it immediately, there will be obtained a new series of terms, called the differences of the terms of the given series. Thus the proposed series being represented by

$$y_0,\quad y_1,\quad y_2,\quad y_3,\ \ldots\ldots\ldots\ldots\ldots\ldots\ y_{n-1},\quad y_n;$$

the series of differences will be

$$y_1-y_0,\quad y_2-y_1,\quad y_3-y_2,\ \ldots\ldots\ldots\ldots\ y_n-y_{n-1};$$

in which y_1-y_0 is the difference of y_0; y_2-y_1, the difference of y_1; y_n-y_{n-1}, the difference of y_{n-1}. These differences are usually denoted by the symbol Δ; the series of differences just given is then written thus

$$\Delta y_0,\quad \Delta y_1,\quad \Delta y_2,\ \ldots\ldots\ldots\ldots\ldots\ldots\ \Delta y_{n-1}.$$

It is to be observed that the number of terms in the series of differences is one less than the number of terms in the proposed series. The same operation can be performed on the series of differences, so as to form the differences of differences, thus

$$\Delta y_1-\Delta y_0,\quad \Delta y_2-\Delta y_1,\ \ldots\ldots\ldots\ \Delta y_{n-1}-\Delta y_{n-2}.$$

These are called the second differences and are usually represented by Δ^2; thus the second differences just given may be written

$$\Delta^2 y_0,\ \Delta^2 y_1,\ \Delta^2 y_2,\ \ldots\ldots\ldots\ldots\ \Delta^2 y_{n-2}.$$

The number of terms in the series of the second differences is two less than the number of terms in the original proposed series. We can form the differences of second differences; these are represented by Δ^3 with the corresponding letter y_0, y_1,.....written on the right of the symbol; the number of terms diminishes by one at each operation. If there are $(n+1)$ terms in the proposed series, there will exist only a single term of the n th differences.

As an example take a series of squares of natural numbers beginning with 5.

Order	0	1	2	3	4	5
x	5	6	7	8	9	10
y	25	36	49	64	81	100
Δy	11	13	15	17	19	
$\Delta^2 y$	2	2	2	2		

The third differences are all zero.

Take a series consisting of cubes of natural numbers, thus

Order	0	1	2	3	4	5	6
x	1	2	3	4	5	6	7
y	1	8	27	64	125	216	343
Δy	7	19	37	61	91	127	
$\Delta^2 y$	12	18	24	30	36		
$\Delta^3 y$	6	6	6	6			

The fourth differences are all zero. All rational integral algebraic

functions of n th degree have their n th differences equal to a constant, so that $(n+1)$th differences are zero.

The successive differences of a transcendental function may be formed in a similar manner.

Differences of a Series of Sines.

Order n	x	$y=\sin x$	Δy	$\Delta^2 y$	$\Delta^3 y$
0	3°	·052336	·017420	−·000020	−·000008
1	4°	·069756	·017400	−·000028	−·000003
2	5°	·087156	·017372	−·000031	−·000006
3	6°	·104528	·017341	−·000037	−·000006
4	7°	·121869	·017304	−·000043	−·000004
5	8°	·139173	·017261	−·000047	−·000006
6	9°	·156434	·017214	−·000053	−·000005
7	10°	·173648	·017161	−·000058	
8	11°	·190809	·017103		
9	12°	·207912			

Differences of a Series of Logarithms.

Order n	x	$y=\log x$	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1·00	·0000 0000	·0043 2137	−·0000 4257	+·0000 0082
1	1·01	·0043 2137	·0042 7880	−·0000 4175	+·0000 0082
2	1·02	·0086 0017	·0042 3705	−·0000 4093	+·0000 0077
3	1·03	·0128 3722	·0041 9612	−·0000 4016	+·0000 0077
4	1·04	·0170 3334	·0041 5596	−·0000 3939	+·0000 0073
5	1·05	·0211 8930	·0041 1657	−·0000 3866	+·0000 0073
6	1·06	·0253 0587	·0040 7791	−·0000 3793	+·0000 0071
7	1·07	·0293 8378	·0040 3998	−·0000 3722	+·0000 0065
8	1·08	·0334 2376	·0040 0276	−·0000 3657	
9	1·09	·0374 2650	·0039 6619		
10	1·10	·0413 9269			

It thus appears that in a series of numbers proceeding according to a regular law, the differences generally become smaller and more nearly equal to one another as their order becomes higher. This general fact is very useful in the construction of a numerical table of any kind.

Laws of combination of the Symbol Δ.

$$\Delta\Big\{\phi(x)+f(x)\Big\}=\Delta\phi(x)+\Delta f(x).$$

Because

$$\begin{aligned}&\Delta\Big\{\phi(x)+f(x)\Big\}\\&=\Big\{\phi(x+h)+f(x+h)\Big\}-\Big\{\phi(x)+f(x)\Big\}\\&=\phi(x+h)-\phi(x)+f(x+h)-f(x)\\&=\Delta\phi(x)+\Delta f(x).\end{aligned}$$

This is the law of *distribution* applied to the symbol Δ.

Let a be a constant and y_x any function of x, then

$$\begin{aligned}\Delta a y_x&=a y_{x+h}-a y_x\\&=a(y_{x+h}-y_x)\\&=a\Delta y_x.\end{aligned}$$

The symbol Δ is therefore commutative with respect to any constant coefficients of the function upon which the operation Δ is applied. This is the law of *commutation.*

The symbol Δ also obeys the *index law,* thus

$$\Delta^m\,\Delta^n\,y_x=\Delta^{m+n}\,y_x,$$

where m and n are positive integers.

If a, b,.....be constants, and u_x, v_x,.....be any functions of x, we have from what have been given

$$\Delta(au_x+bv_x+\ldots\ldots\ldots)=a\Delta u_x+b\Delta v_x+\ldots\ldots\ldots\ldots.$$

Theorem $\Delta(u_x v_x) = u_x \Delta v_x + v_x \Delta u_x + \Delta u_x \times \Delta v_x.$

For,
$$\begin{aligned}\Delta(u_x v_x) &= u_{x+h} v_{x+h} - u_x v_x \\ &= u_{x+h} v_{x+h} - u_{x+h} v_x + u_{x+h} v_x - u_x v_x \\ &= u_{x+h} \Delta v_x + v_x \Delta u_x \\ &= (u_x + \Delta u_x) \Delta v_x + v_x \Delta u_x \\ &= u_x \Delta v_x + v_x \Delta u_x + \Delta u_x \Delta v_x.\end{aligned}$$

This result may be written in the following form :—

$$\Delta(u_x v_x) = u_x v_x \left(\frac{\Delta u_x}{u_x} + \frac{\Delta v_x}{v_x} + \frac{\Delta u_x}{u_x} \frac{\Delta v_x}{v_x} \right).$$

To expand y_{x+n} in a series consisting of y_x and the successive differences.

Since $$\Delta y_x = y_{x+1} - y_x,$$

in which the difference of the independent variable is made equal to 1, we have

$$y_x + \Delta y_x = y_{x+1}$$

or $$(1 + \Delta) y_x = y_{x+1},$$

where $(1 + \Delta)$ may be regarded as a single operator, whose effect is to increase the independent variable x contained in the function by 1.

Therefore
$$\begin{aligned}(1 + \Delta)(1 + \Delta) y_x &= (1 + \Delta) y_{x+1} \\ &= y_{x+2},\end{aligned}$$

i. e., $$(1 + \Delta)^2 y_x = y_{x+2}.$$

Similarly $$(1 + \Delta)^3 y_x = y_{x+3}$$

and generally

$$(1 + \Delta)^n y_x = y_{x+n}.$$

Expanding the left hand member by the binomial theorem, we have

$$\left\{1+n\Delta+\frac{1}{\lfloor 2}n(n-1)\Delta^2+\frac{1}{\lfloor 3}n(n-1)(n-2)\Delta^3+\ldots\ldots\right\}y_x=y_{x+n}.$$

Hence we have

$$y_{x+n}=y_x+n\Delta y_x+\frac{1}{\lfloor 2}n(n-1)\Delta^2y_x$$
$$+\frac{1}{\lfloor 3}n(n-1)(n-2)\Delta^3y_x+\ldots\ldots\quad\ldots\ldots\ldots\quad\text{(A)}$$

To express $\Delta^n y_x$ in terms y_x and its successive values.

$$\Delta^n y_x=\left\{(1+\Delta)-1\right\}^n y_x$$
$$=\left\{(1+\Delta)^n-n(1+\Delta)^{n-1}+\frac{1}{\lfloor 2}n(n-1)(1+\Delta)^{n-2}\right.$$
$$\left.-\frac{1}{\lfloor 3}n(n-1)(n-2)(1+\Delta)^{n-3}+\ldots\ldots\right\}y_x.$$

Hence we have

$$\Delta^n y_x=y_{x+n}-ny_{x+n-1}+\frac{1}{\lfloor 2}n(n-1)y_{x+n-2}$$
$$-\frac{1}{\lfloor 3}n(n-1)(n-2)y_{x+n-3}+\ldots\ldots\quad\ldots\ldots\quad\text{(B)}$$

The above equations (A) and (B) are the most well known formulæ of interpolation.

Example. To find the sum of cubes of natural numbers.

Let the required sum be

$$y_n=1^3+2^3+3^3+\ldots\ldots\ldots\ldots+(n-1)^3+n^3.$$

The suffix x in the equation (A) being 0 in this case, we have

$$\Delta y_n=y_{n+1}-y_n=(n+1)^3,$$

$$\begin{aligned}\Delta^2 y_n &= (n+2)^3-(n+1)^3\\ &= 3n^2+9n+7,\\ \Delta^3 y_n &= 3(n+1)^2+9(n+1)+7-(3n^2+9n+7)\\ &= 6n+12,\\ \Delta^4 y_n &= 6(n+1)+12-(6n+12)\\ &= 6,\end{aligned}$$

and the differences of higher orders are all zero. Putting $n=0$, we obtain

$$y_0=0, \qquad \Delta y_0=1,$$
$$\Delta^2 y_0=7, \qquad \Delta^3 y_0=12,$$
$$\Delta^4 y_0=6,$$

and the equation (A) takes the form :—

$$y_n=y_0+n\Delta y_0+\frac{1}{2}n(n-1)\Delta^2 y_0+\frac{1}{6}n(n-1)(n-2)\Delta^3 y_0 + \frac{1}{24}n(n-1)(n-2)(n-3)\Delta^4 y_0+\ldots\ldots$$

Substituting the values of the successive differences in this last equation, we obtain

$$\begin{aligned}y_n &= 0+n+\frac{1}{2}n(n-1)7+\frac{1}{6}n(n-1)(n-2)12\\ &\qquad +\frac{1}{24}n(n-1)(n-2)(n-3)6\\ &= \left\{\frac{1}{2}n(n-1)\right\}^2.\end{aligned}$$

Although in obtaining the formula (A), it has been understood that the number n is a positive integer, the result can be applied to cases in which n is either negative, or fractional, or negative and fractional, so long as the function is continuous over the range considered. This extension may be regarded as one example of the application of the principle of continuity.

Example. From a table of saturated steam, we find the following values of the absolute pressures in lbs. per sq. inch and the corresponding temperatures in degrees Fah.

Pressure in lbs. per sq. inch	45,	50,	55,	60;
Temperature in Fah. scale	274·4,	281·0,	287·1,	292·7.

Required to find the temperatures corresponding to pressures of 65 and 57 lbs. per sq. inch.

Solution:—We first form a table of difference as follows:—

n	0	1	2	3
p	45	50	55	60
T	274·4	281·0	287·1	292·7
ΔT	6·6	6·1	5·6	
$\Delta^2 T$	−0·5	−0·5		
$\Delta^3 T$	0			

For the pressure of 65 lbs. per sq. inch, the value of n is 4 and from formula (A) we have

$$T_4 = 274{\cdot}4 + 4 \times 6{\cdot}6 + \frac{1}{2} \times 4 \times 3 \times (-0{\cdot}5)$$

$$= 297{\cdot}8° \text{ Fah.}$$

For the pressure of 57 lbs. per sq. inch, the value of $n = (57 - 45) \div$ common difference $5 = 2.4$ and formula (A) gives

$$T_{2.4} = 274{\cdot}4 + 2{\cdot}4 \times 6{\cdot}6 + \frac{1}{2} \times 2{\cdot}4 \times 1{\cdot}4 \times (-0{\cdot}5)$$

$$= 289{\cdot}6° \text{ Fah.}$$

Example. Given the expectation* of life of a person for the ages of 10, 15, 20, and 30 years equal to 45·46, 41·79, 38·31, and 31·75 years respectively, to find the expectation of life for the age of 25 years. These data are taken from 生命保險論 (On Mortality and Life Insurance) by Prof. R. Huzisawa of the Science College, Tokyo Imperial University.

It is a remarkable fact that the successive differences of any continuous function diminish in general very rapidly and consequently any function expanded in a series of successive differences converges very rapidly after a few first terms. The left hand side of the formula (B) may therefore be put equal to zero for values of n equal to a small number. The successive differential coefficients of a continuous function do not manifest any general law of variation such as is stated above, in regards to their numerical values.

In the example just now proposed, the successive values of the function correspond to the ages of 10, 15, 20, and 30 years. These ages proceed by an equal difference of 5 years with a gap between 20 and 30 years. Hence using formula (B) and putting

$$\Delta^4 y_x = 0 \qquad \text{and} \qquad x = 0,$$

we have

$$0 = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0,$$

therefore
$$y_3 = \frac{1}{4}(y_4 + 6y_2 + y_0 - 4y_1)$$

$$= \frac{1}{4}(31{\cdot}75 + 6 \times 38{\cdot}31 + 45{\cdot}46 - 4 \times 41{\cdot}79)$$

$$= 34{\cdot}98 \text{ years.}$$

In the table given by Prof. R. Huzisawa, the value of y_3 is 34·99 years.

* Average after-life time.

Newton's Method of Approximation to the Numerical Value of a Root of an Equation.

Suppose we require a root of an equation

$$f(x)=0,$$

where $f(x)$ is any function of x. An approximation to the required root can always be determined roughly by plotting a few points, and drawing the curve $y=f(x)$. The abscissa of the point of intersection of the curve with the axis of x is an approximate root required.

Let a be an approximate value of the root thus found out and $a+h$ the true root. Then

$$f(a+h)=0.$$

But by Taylor's Theorem

$$f(a+h)=f(a)+hf'(a)+\frac{1}{\lfloor 2}h^2f''(a)+\ldots\ldots\ldots$$

Since h is a small quantity, we have

$$f(a+h)=f(a)+hf'(a) \text{ nearly.}$$

But the left hand member is equal to zero and therefore

$$f(a)+hf'(a)=0,$$

from which

$$h=-\frac{f(a)}{f'(a)}.$$

Thus a nearer root is

$$a+h=a-\frac{f(a)}{f'(a)}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(C)}$$

Example. Required the root of the equation

$$x^x=100.$$

Writing the equation in the form

$$y=x^x-100,$$

we have

when	$x=1,$	$y=-99;$
„	$x=2,$	$y=-96;$
„	$x=3,$	$y=-73;$
„	$x=4,$	$y=+156;$

Thus we see that there exists a root between $x=3$ and 4. Take the common logarithms of both members of the given equation and then putting $y=x\log_{10}x-2$, we have

when	$x=1,$	$y=-2;$
„	$x=2,$	$y=-1{\cdot}40;$
„	$x=3,$	$y=-0{\cdot}57;$
„	$x=4,$	$y=+0{\cdot}41.$

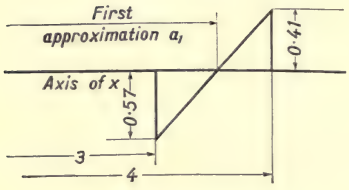


Fig. 1.

$$\frac{dy}{dx}=\log_{10}x+\log_{10}e.$$

Referring to Fig. 1, a very rough estimate for the root of the equation may be found as follows:—

First approximation $$a_1=3+\frac{4-3}{\cdot57+\cdot41}\times\cdot57$$

$$=3{\cdot}58.$$

Taking, then, 3·6 as a first approximation, we have for a second approximation

$$a_2=a_1+h_1=a_1-\frac{f(a_1)}{f'(a_1)}$$

$$=3.6-\frac{3\cdot6\times\log 3\cdot6-2}{\log 3\cdot6+\cdot43429}$$

$$=3\cdot5973,$$

which is correct to the last figure.

Example. Into the hollow of a round hollow shaft fits a solid shaft of the same material. The two shafts are required to be of the same strength. Find the outer diameter of the hollow shaft in terms of the diameter of the solid shaft.

Solution:—Let $D=$ the outer diameter of the hollow shaft, $d=$ the diameter of the hollow or of the solid shaft, $f=$ the safe shearing stress of the material. Since the two shafts are to be of the same strength, we have

$$f\frac{\pi}{16}\frac{D^4-d^4}{D}=f\frac{\pi}{16}d^3,$$

from which we have

$$x^4-x-1=0,$$

where x stands for the ratio $D\div d$. Now writing the equation in the form

$$y=x^4-x-1,$$

we have,

when	$x=1\cdot0,$	$y=-1\cdot00;$
„	$x=1\cdot2,$	$y=-0\cdot13;$
„	$x=1\cdot4,$	$y=+1\cdot44.$

$x=1\cdot2$ may therefore be taken as a first approximation to the root. A second approximate root is then

$$a+h=a-\frac{f(a)}{f'(a)}$$

$$= a - \frac{a^4 - a - 1}{4a^3 - 1} = 1{\cdot}2 - \frac{(1{\cdot}2)^4 - 1{\cdot}2 - 1}{4(1{\cdot}2)^3 - 1}$$

$$= 1{\cdot}221.$$

Thus the outer diameter of the hollow shaft is 1·221 times the diameter of the solid shaft.

The degree of accuracy of the result can be estimated by comparing it with the next approximation. Thus a third approximation to the root in this example being $x = 1{\cdot}2208$, the above result is correct to the last figure.

Example. To find the value of θ satisfying the equation $\theta = \tan\theta$ where θ is in radians.

Solution :—Let $f(\theta) = \tan\theta - \theta$. Then by means of tables of radian measures and trigonometrical tangents, we find

when $\theta = 180° = 3{\cdot}1416$ radians, $f(\theta) = -3{\cdot}1416$;

,, $\theta = 180° + 60° = 4{\cdot}1888$, $f(\theta) = -2{\cdot}4567$;

,, $\theta = 180° + 70° = 4{\cdot}3633$, $f(\theta) = -1{\cdot}6158$;

,, $\theta = 180° + 80° = 4{\cdot}5397$, $f(\theta) = +1{\cdot}1334$.

By using the last two sets of values of θ and $f(\theta)$ and assuming in the mean time a straight line variation of $f(\theta)$ with θ, as in Fig. 1, we find an approximate root to be

$$\theta_1 = 180° + 76° = 3{\cdot}1416 + 1{\cdot}3265 = 4{\cdot}4681 \quad \text{radians},$$

$$f(\theta_1) = 4{\cdot}0108 - 4{\cdot}4681 = -{\cdot}4573.$$

$$f'(\theta) = \sec^2\theta - 1 = \tan^2\theta.$$

A second approximate root is

$$\theta_2 = \theta_1 - \frac{\tan\theta_1 - \theta_1}{(\tan\theta_1)^2}$$

$$= 4{\cdot}4681 - \frac{-{\cdot}4573}{(4{\cdot}0108)^2} = 4{\cdot}4965.$$

A third approximation is

$$\theta_3 = 4{\cdot}4965 - \frac{4{\cdot}5609 - 4{\cdot}4965}{(4{\cdot}5609)^2}$$

$$= 4{\cdot}4934$$

$$= 3{\cdot}1416 + 1{\cdot}3518 = 180° + 77°27',$$

which is correct to five significant figures. A more exact value of the angle is 4·493409 radians

$$= 180° + 77°27'12{\cdot}2''.$$

The solution of this equation is required in establishing Euler's formula for long columns rounded at one end and flat at the other, and also in finding the height of the level of water flowing through a circular pipe partially filled so that the velocity of flow may be a maximum.

The equation $x^x = 100$ can be solved by the theory of differences. Putting the equation in the form

$$y_x = x \log x - 2,$$

we obtain the following table :—

n	0	1	2	3
x	3.4	3·5	3·6	3·7
y_x	−·192968	−·095755	·002680	·102340
Δy_x	·097213	·098435	·099660	
$\Delta^2 y_x$	·001222	·001225		
$\Delta^3 y_x$	·000003			

Neglecting the third and higher differences, the formula (A) becomes

$$y_{3\cdot4+n}=y_{3\cdot4}+n\Delta y_{3\cdot4}+\frac{1}{2}n(n-1)\Delta^2 y_{3\cdot4}.$$

Since the value of $y_{3\cdot4+n}$ must be zero for value of x satisfying the given equation, the above equation becomes

$$0=-\cdot192968+n\times\cdot097213+\frac{1}{2}n(n-1)\times\cdot001222.$$

Solving this for n, we obtain

$$n=-79\cdot05\pm81\cdot0229$$

$$=1\cdot9729$$

which alone is admissible for the question, and therefore

$$x=3\cdot4+\cdot1\times1\cdot9729=3\cdot59729,$$

the correct value being 3·597285.

A General Principle. *Successive differences of a continuous function diminish in general very rapidly. When this is not the case, the differences of logarithms of such a function diminish rapidly.* ..... (D)

In an extended series of numerical values of a continuous function, it may sometimes happen that the successive differences do not diminish so rapidly as is desirable, or that they tend first to diminish and afterwards to increase. In such a case the law of variation of the function is higher and more general than rational and integral algebraic law, which the method of interpolation assumes to follow at least approximately. Now it is chiefly for a small range of an independent variable that a transcendental function can be very well approximated by a rational and integral algebraic expression. By a previous mathematical treatment, however, an unwieldy function can be greatly reduced in magnitude and approximated very well over an ex-

tended range of the variable by an algebraic expression. This is the taking of the logarithm of the function. For easy handling and transportation of bulky stuff such as cotton, a previous mechanical treatment under an hydraulic press is necessary. In this sense the logarithm may be called a mathematical press for unwieldy functions.

Example. Regnault gives the following data as a conclusion from his experiments on saturated steam for temperatures ranging from $100°C.$ to $220°C.$:—

Temperature in degrees centigrade	Pressure in mm. of mercury
100	760·00
130	2030·0
160	4651·6
190	9426
220	17390

Suppose that the pressure corresponding to $150°C.$ is required.

Solution:—Here the temperatures are given at a constant difference of $30°C.$ and consequently the general formula (A) seems to be applicable.

The successive differences are found to be as follows:—

n	0	1	2	3	4
t	100	130	160	190	220
p	760·0	2030·0	4651·6	9426·0	17390·0
Δp	1270·0	2621·6	4774·4	7964·0	
$\Delta^2 p$	1351·6	2152·8	3189·6		
$\Delta^3 p$	801·2	1036·8			
$\Delta^4 p$	235·6				

But here the diminution of successive differences is so very slow that we are not at all justified to neglect the fifth differences. Hence in accordance with the general principle (D) we take the logarithms of the pressures instead of the pressures themselves. The fourth difference of the logarithms of the pressures is very small so that the fifth can safely be neglected; this is seen from the following table.

n	0	1	2	3	4
t	100	130	160	190	220
$\log p$	2·88081	3·30750	3·66760	3·97433	4·24030
$\Delta \log p$	·42669	·36010	·30673	·26597	
$\Delta^2 \log p$	−·06659	−·05337	−·04076		
$\Delta^3 \log p$	·01322	·01261			
$\Delta^4 \log p$	−·00061				

The value of n corresponding to $150°C.$ is $(150-100)\div 30=\frac{5}{3}$. Now from the formula (A), we have

$$\log (p)_{x+n}=2\cdot88081+\frac{5}{3}\times\cdot42669+\frac{1}{2}\times\frac{5}{3}\times\frac{2}{3}\times(-\cdot06659)$$

$$+\frac{1}{6}\times\frac{5}{3}\times\frac{2}{3}\times\left(-\frac{1}{3}\right)\times\cdot01322$$

$$+\frac{1}{24}\times\frac{5}{3}\times\frac{2}{3}\times\left(-\frac{1}{3}\right)\times\left(-\frac{5}{3}\right)\times\cdot00061$$

$$=3\cdot55417=\log(3582\cdot3).$$

Thus the required pressure is 3582·3 mm. of mercury. From Prof. C. H. Peabody's Table of Saturated Steam, we find that a more correct value is 3582·0 mm.

Suppose now that it is required **to form an equation giving the relation between the temperature and the pressure of saturated steam** within the range of temperatures given by Regnault. Neglecting the differences of the fourth and higher orders, we have

$$\log p = 2{\cdot}88081 + x(\cdot 42669)$$
$$+ \frac{1}{2}x(x-1)(-\cdot 06659)$$
$$+ \frac{1}{6}x(x-1)(x-2)(\cdot 01322)$$
$$= 2{\cdot}88081 + 0{\cdot}46439x - 0{\cdot}039905x^2 + 0{\cdot}002203x^3,$$

in which

$$x = \frac{t-100}{30}.$$

From this formula we find

Temperature t in C.	Pressure p by Regnault mm.	Pressure p by our formula mm.	Error of our formula
100	760·00	760·0	0
130	2030·0	2030·0	0
160	4651·6	4652·6	+1·0
190	9426	9427·9	+1·9
220	17390	17413	+23

A formula better than the above can be obtained by the method of Least Squares and is as follows:—

$$\log p = 2{\cdot}88081 + 0{\cdot}46418x - 0{\cdot}039688x^2 + 0{\cdot}002153x^3,$$

which gives the following values of the pressure:—

x	t in °C.	p in mm. of mercury	*dif.*
0	100	760·00	0
1	130	2029·8	−0·2
2	160	4652·0	+0·4
3	190	9425·2	−0·8
4	220	17390·	0

Example. From Prof. C. H. Peabody's Steam Tables are obtained the following data relating to saturated vapour of ether:—

Temperature in degrees Centigrade t	Pressure in mm. of mercury p
0	184·39
30	634·80
60	1725·0
90	3898·3
120	7719·2

Let it be required to find an equation expressing the relation between the t and p.

Solution:—The successive differences of the pressure stand as follows:—

n	0	1	2	3	4
t	0	30	60	90	120
p	184·39	634·80	1725·0	3898·3	7719·2
Δp	450·4	1090·2	2173·3	3820·9	
$\Delta^2 p$	640·2	1083·1	1647·6		
$\Delta^3 p$	442·9	564·5			
$\Delta^4 p$	121·6				

The diminution of the differences being very slow, the fourth difference can not be neglected. We therefore use logarithms as shown below.

n	0	1	2	3	4
t	0	30	60	90	120
$\log p$	2·26574	2·80264	3·23679	3·59088	3·88758
$\Delta \log p$	·53690	·43415	·35409	·29669	
$\Delta^2 \log p$	−·10275	−·08006	−·05740		
$\Delta^3 \log p$	+·02269	+·02266			
$\Delta^4 \log p$	−·00003				

Neglecting the difference of the fourth order, we have

$$\log p = 2{\cdot}26574 + x \times {\cdot}53690 + \frac{1}{2}x(x-1) \times (-{\cdot}10275)$$

$$+ \frac{1}{6}x(x-1)(x-2) \times {\cdot}02269,$$

$$= 2{\cdot}26574 + {\cdot}59584x - {\cdot}062720x^2 + {\cdot}0037817x^3,$$

in which $x = t \div 30$. For t equal to 0, 30, 60, and 90 degrees, this equation gives exactly the same values of p as in the data. For $t = 120°\,C$, p will be found equal to 7719·7 mm. of mercury.

The rules and formulæ thus far considered are derived on the supposition that the series of differences proceeds by constant differences, whatever may be the magnitude of that constant. When this is not the case, the formulæ (A) and (B) are not applicable.

Problem. Given the values y_a, y_b, y_c,.....of a function corresponding to $x=a$, b, c,.....respectively, which are not consecutive and equidistant, to find any other value of y corresponding to x.

The general expression being assumed to be rational and integral, the following form of equation, known as **Lagrange's Theorem for Interpolation,** enables us to write down at once the values of the constants involved.

$$\begin{aligned} y = {} & A_0(x-b)(x-c)(x-d).....(x-k) \\ & + A_1(x-a)(x-c)(x-d).....(x-k) \\ & + A_2(x-a)(x-b)(x-d).....(x-k) \\ & + A_3(x-a)(x-b)(x-c).....(x-k) \\ & + ...\text{to } n \text{ terms,} \end{aligned}$$

n being the number of values of the given function, and y_k the last term corresponding to $x=k$. Making $x=a$ and $y=y_a$, all the terms on the right hand side of the above equation vanish except the first and we obtain

$$\left.\begin{aligned} A_0 &= \frac{y_a}{(a-b)(a-c)(a-d).....(a-k)}, \\ \text{Similarly} \\ A_1 &= \frac{y_b}{(b-a)(b-c)(b-d).....(b-k)}, \\ A_2 &= \frac{y_c}{(c-a)(c-b)(c-d).....(c-k)}, \\ &\&c. \quad \&c. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ (E)$$

Newton's Interpolation formula. Here also the general expression is assumed to be rational and integral, but the form of the equation is different, being

$$y=A+B(x-a)+C(x-a)(x-b)+D(x-a)(x-b)(x-c)$$
$$+E(x-a)(x-b)(x-c)(x-d)+\ldots\ldots\ldots\ldots$$

If $x=a$, then $y_a=A$. Put this value of A in the equation and then make $x=b$ so that $y=y_b$, then $y_b=y_a+B(b-c)$,

therefore $$B=\frac{y_b-y_a}{b-a},$$

which may be written $$\left(\frac{\Delta}{\Delta x}\right)y_{ba}.$$

Substitute this in the assumed equation and make $x=c$ so that $y=y_c$, then we have

$$y_c=y_a+\left(\frac{\Delta}{\Delta x}\right)y_{ba}(c-a)+C(c-a)(c-b),$$

from which we get

$$\frac{y_c-y_a}{c-a}=\left(\frac{\Delta}{\Delta x}\right)y_{ba}+C(c-b),$$

that is,

$$\left(\frac{\Delta}{\Delta x}\right)y_{ca}-\left(\frac{\Delta}{\Delta x}\right)y_{ba}=C(c-b).$$

Therefore $$C=\frac{\left(\frac{\Delta}{\Delta x}\right)y_{ca}-\left(\frac{\Delta}{\Delta x}\right)y_{ba}}{c-b},$$

which may be written $$\left(\frac{\Delta}{\Delta x}\right)^2 y_{cb}.$$

These are called the difference coefficients and are quite analogous to differential coefficients.

Newton's formula of interporlation may now be written

$$y=y_a+(x-a)\left(\frac{\Delta}{\Delta x}\right)y_{ba}+(x-a)(x-b)\left(\frac{\Delta}{\Delta x}\right)^2 y_{cb}$$
$$+(x-a)(x-b)(x-c)\left(\frac{\Delta}{\Delta x}\right)^3 y_{dc}+\ldots\ldots\ldots\ldots\ldots\ldots \quad (F)$$

The successive values of the difference coefficients in Newton's formula may be calculated most conveniently in a tabular form as follows :—

x	a	b	c	d	e	f
y	y_a	y_b	y_c	y_d	y_e	y_f
$\left(\frac{\Delta}{\Delta x}\right) y$	$_{ba}$	$_{ca}$	$_{da}$	$_{ea}$	$_{fa}$	
$\left(\frac{\Delta}{\Delta x}\right)^2 y$	$_{cb}$	$_{db}$	$_{eb}$	$_{fb}$		
$\left(\frac{\Delta}{\Delta x}\right)^3 y$	$_{dc}$	$_{ec}$	$_{fc}$			
$\left(\frac{\Delta}{\Delta x}\right)^4 y$	$_{ed}$	$_{fd}$				
$\left(\frac{\Delta}{\Delta x}\right)^5 y$	$_{fe}$					

Any of the terms in the Table, as for example the one corresponding to the intersection of 5th line and 3rd column is

$$\left(\frac{\Delta}{\Delta x}\right)^3 y_{ec},$$

which indicates the operation

$$\frac{\left(\frac{\Delta}{\Delta x}\right)^2 y_{eb} - \left(\frac{\Delta}{\Delta x}\right)^2 y_{cb}}{e - c}.$$

Example. Given $\tan 86°0' = 14{\cdot}30$, $\tan 86°18 = 15{\cdot}46$, $\tan 86°30' = 16{\cdot}35$, $\tan 86°36' = 16{\cdot}83$; to find an approximate value of $\tan 86°54'$

Solution :—

x	$a=86°0'$	$b=86°18'$	$c=86°30'$	$d=86°36'$
y	$y_a=14{\cdot}30$	$y_b=15{\cdot}46$	$y_c=16{\cdot}35$	$y_d=16{\cdot}83$
$\left(\frac{\Delta}{\Delta x}\right)y$	·0644 $_{ba}$	·0683 $_{ca}$	·0703 $_{da}$	
$\left(\frac{\Delta}{\Delta x}\right)^2 y$	·000325 $_{cb}$	·000333 $_{db}$		
$\left(\frac{\Delta}{\Delta x}\right)^3 y$	·00000133 $_{dc}$			

Using Newton's formula of interpolation, the value of x corresponding to the required value of y is $86°54'$. Thus we have

$$x-a=86°54'-86°0'=54,$$

$$(x-a)(x-b)=54(86°54'-86°18')=54\times 36=1944,$$

$$(x-a)(x-b)(x-c)=1944(86°54'-86°30')=1944\times 24=46656.$$

Therefore $y=14{\cdot}30+54\times{\cdot}0644+1944\times{\cdot}000325+46656\times{\cdot}00000133$

$=18{\cdot}47.$

The exact value of $\tan 86°54'$ to two places of decimals is 18·46.

Approximate value of the Integral of a Continuous Function.

Definition :—

$$\Delta y_a=y_{a+h}-y_a.$$

Then

$$y_a+\Delta y_a=y_{a+h},$$

which may be written symbolicaly thus

$$(1+\Delta)y_a = y_{a+h}.$$

Hence

$$(1+\Delta)^2 y_a = (1+\Delta)y_{a+h} = y_{a+2h},$$

similarly

$$(1+\Delta)^3 y_a = y_{a+3h}$$

and generally

$$(1+\Delta)^n y_a = y_{a+nh}.$$

Expanding the left hand member by the binomial theorem, we have

$$y_{a+nh} = \left\{1 + n\Delta + \frac{1}{\lfloor 2}n(n-1)\Delta^2 + \frac{1}{\lfloor 3}n(n-1)(n-2)\Delta^3 + \ldots\ldots\right\} y_a$$

$$= y_a + n\Delta y_a + \frac{1}{\lfloor 2}n(n-1)\Delta^2 y_a + \frac{1}{\lfloor 3}n(n-1)(n-2)\Delta^3 y_a + \ldots\ldots \quad (A_1)$$

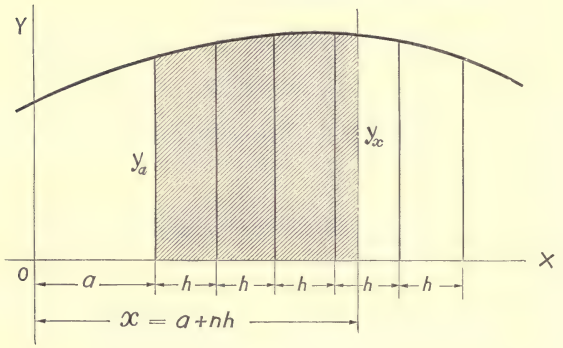


Fig. 2.

Referring to Fig. 2 and considering a and h as constants, the above equation gives the value of the function y corresponding to a positive integral values of n and so long as $n+1$ is not greater than the number of terms of the given function y, the

value obtained by the equation is exact. Now the equation being in rational algebraic positive integral powers of n, the value of y corresponding to any fractional value of n must be continuous over two consecutive integral values of n. Hence when the form of the functionality is continuous and the successive differences are not large, the value of y obtained by giving a fractional value to n cannot much differ from the true value. We are therefore quite safe to assume that n in the above equation is an independent variable capable of assuming any fractional or integral value, a and h being constants. The variable abscissa x is evidently connected with n by the relation

$$x=a+nh. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

The value of the integral of y between $x=a$ and x is

$$\int_a^x y\,dx = y_a\int_a^x dx + \Delta y_a\int_a^x n\,dx + \frac{1}{\underline{|2}}\Delta^2 y_a\int_a^x n(n-1)dx + \frac{1}{\underline{|3}}\Delta^3 y_a\int_a^x n(n-1)(n-2)dx + \ldots\ldots .$$

But from (1), we have $dx=h\,dn$ so that when $x=a$, $n=0$. Effecting the integration, we have

$$\int_a^x y\,dx = h\Big\{ny_a + \frac{n^2}{2}\Delta y_a + \Big(\frac{n^3}{3}-\frac{n^2}{2}\Big)\frac{\Delta^2 y}{\underline{|2}} + \Big(\frac{n^4}{4}-n^3+n^2\Big)\frac{\Delta^3 y}{\underline{|3}} + \Big(\frac{n^5}{5}-\frac{3n^4}{2}+\frac{11n^3}{3}-3n^2\Big)\frac{\Delta^4 y}{\underline{|4}} + \Big(\frac{n^6}{6}-2n^5+\frac{35}{4}n^4-\frac{50}{3}n^3+12\,n^2\Big)\frac{\Delta^5 y}{\underline{|5}} + \ldots\ldots\ldots\ldots\Big\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (G)$$

Neglecting $\Delta^3 y_a$ and all the succeeding differences is equivalent to approximating the integral by the area of a parabola of the second degree. Assume then $\Delta^3 y_a = 0$ and put $\Delta y_a = y_{a+h} - y_a$, $\Delta^2 y_a = y_{a+2h} - 2y_{a+h} + y_a$. The general equation (G) then becomes

$$\int_a^x y\,dx = \frac{nh}{12}\Big\{(12 - 9n + 2n^2)y_0 + 4n(3-n)y_1 - n(3-2n)y_2\Big\}, \quad \ldots\ldots\ldots \quad (g_1)$$

in which y_0, y_1, y_2 are put for y_a, y_{a+h}, y_{a+2h} for simplicity. This integral is shown by Fig. 3.

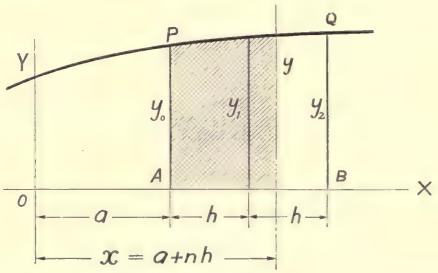


Fig. 3.

Making $n = \frac{1}{2}$, we obtain the area corresponding to $\frac{1}{2}$th of the whole abscissa AB, thus :—

$$\text{Area} = \frac{(\frac{1}{2}h)}{12}\Big(8y_0 + 5y_1 - y_2\Big),$$

which may be called $8-5$ rule.

Making $n = 1$, we obtain the area corresponding to one half of abscissa AB, thus :—

$$\text{Area} = \frac{h}{12}\Big(5y_0 + 8y_1 - y_2\Big),$$

which is a well known $5-8$ rule.

Make $n = 1\frac{1}{2}$ and the area corresponding to $\frac{3}{4}$ of the abscissa is obtained

$$\text{Area} = \frac{(1\frac{1}{2}h)}{4}\Big(y_0 + 3y_1\Big),$$

which is a singular result, the third ordinate disappearing altogether.

Making $n=2$, we have

$$\text{Area} = \frac{2h}{6}\Big(y_0 + 4y_1 + y_2\Big),$$

which is Simpson's first rule.

Making $n=2\frac{1}{2}$, we have

$$\text{Area} = \frac{(2\frac{1}{2}h)}{12}\Big(2y_0 + 5y_1 + 5y_2\Big).$$

Make $n=3$ and we obtain

$$\text{Area} = \frac{3h}{4}\Big(y_0 + 3y_2\Big),$$

corresponding to $1\frac{1}{2}$ times the abscissa AB. Here the second ordinate y_1 disappears altogether. In all these cases, the coefficients of y are related to the divisor, thus:—

$$\frac{8+5-1}{12}=1, \quad \frac{5+8-1}{12}=1, \quad \frac{1+3}{4}=1,$$

$$\frac{1+4+1}{6}=1, \quad \frac{2+5+5}{12}=1, \quad \frac{1+3}{4}=1.$$

Neglect $\Delta^4 y$ and all the successive higher differences. This is equivalent to taking 4 equidistant ordinates and approximating to the value of the integral by means of the parabola of the 3rd degree.

Putting $\Delta^4 y=0$ and substituting

$$\Delta y = y_1 - y_0, \qquad \Delta^2 y = y_2 - 2y_1 + y_0,$$

and

$$\Delta^3 y = y_3 - 3y_2 + 3y_1 - y_0,$$

we obtain

$$\int_a^x y\,dx = \frac{nh}{24}\Big\{(4-n)(6-4n+n^2)y_0 + n(36-20n+3n^2)y_1 - n(18-16n+3n^2)y_2 + n(2-n)^2 y_3\Big\}. \quad \ldots\ldots\ldots \quad (g_2)$$

Let $n=1$, then the integral $= \frac{h}{24}(9y_0+19y_1-5y_2+y_3)$.

Let $n=2$, then the integral $= \frac{2h}{12}(y_0+4y_1+y_2)$,

which is Simpson's 1st rule.

Let $n=3$, then the integral $= \frac{3h}{8}(y_0+3y_1+3y_2+y_3)$,

which is Simpson's 2nd rule.

Let $n=4$, then the integral $= \frac{4h}{3}(2y_1-y_2+2y_3)$,

in which the first ordinate does not appear. The writer believes that the two formulæ (g_1) and (g_2) are given for the first time.

Another useful formula for **an approximate value of an integral first given by Laplace is**

$$\int_{x_1}^{x_2} y\,dx = \frac{x_2-x_1}{n}\Big\{\frac{y_0}{2} + y_1 + y_2 + y_3 + \ldots\ldots\ldots + y_{n-1} + \frac{y_n}{2} - \frac{1}{12}(\Delta y_{n-1} - \Delta y_0) - \frac{1}{24}(\Delta^2 y_{n-2} + \Delta^2 y_0) - \ldots\ldots\ldots\Big\}.$$

To demonstrate this formula of Laplace. We have seen that

$$(1+\Delta)y_x = y_{x+1}.$$

But by Taylor's theorem

$$y_{x+1} = y_x + \frac{d}{dx}y_x + \frac{1}{\underline{|2}}\frac{d^2}{dx^2}y_x + \frac{1}{\underline{|3}}\frac{d^3}{dx^3}y_x + \ldots\ldots\ldots$$

$$= \Big\{1 + \frac{d}{dx} + \frac{1}{\underline{|2}}\Big(\frac{d}{dx}\Big)^2 + \frac{1}{\underline{|3}}\Big(\frac{d}{dx}\Big)^3 + \ldots\ldots\ldots\Big\}y_x$$

$$= e^{\frac{d}{dx}}y_x.$$

Therefore $(1+\Delta)y_x = e^{\frac{d}{dx}} y_x$

and therefore $1+\Delta = e^{\frac{d}{dx}}$, ... (H)

so that

$$\frac{d}{dx} = \log(1+\Delta).$$

Now

$$\Delta = \frac{d}{dx} \times \frac{\Delta}{\frac{d}{dx}} = \frac{d}{dx} \times \frac{\Delta}{\log(1+\Delta)}$$

$$= \frac{d}{dx} \times \left\{ 1 + \frac{\Delta}{2} - \frac{1}{12}\Delta^2 + \frac{1}{24}\Delta^3 - \frac{19}{720}\Delta^4 + \frac{3}{160}\Delta^5 - \ldots\ldots \right\}^* \quad \ldots\ldots\ldots\ldots \quad (I)$$

$$= \frac{d}{dx} \times \phi(\Delta), \text{ say.}$$

Hence

$$\Delta w_x = \frac{d}{dx}\phi(\Delta) w_x$$

where w is any function of x. Now let $y_x = \Delta w_x$,

then $$\int_r^{r+1} y_x dx = \int_r^{r+1} \Delta w_x dx = \int_r^{r+1} \frac{d}{dx}\phi(\Delta) w_x dx$$

---

* This expansion can be obtained by Maclaurin's theorem, or as follows:—

$$\frac{x}{\log(1+x)} = \frac{x}{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \ldots\ldots}.$$

Assume this to be $A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \ldots\ldots$. Clear of the fraction and equate the coefficients of all powers of x to zero, thus:—

$$A_0 = 1,$$
$$A_1 - \tfrac{1}{2}A_0 = 0,$$
$$A_2 - \tfrac{1}{2}A_1 + \tfrac{1}{3}A_0 = 0,$$
$$A_3 - \tfrac{1}{2}A_2 + \tfrac{1}{3}A_1 - \tfrac{1}{4}A_0 = 0,$$
$$A_4 - \tfrac{1}{2}A_3 + \tfrac{1}{3}A_2 - \tfrac{1}{4}A_1 + \tfrac{1}{5}A_0 = 0,$$
$$A_5 - \tfrac{1}{2}A_4 + \tfrac{1}{3}A_3 - \tfrac{1}{4}A_2 + \tfrac{1}{5}A_1 - \tfrac{1}{6}A_0 = 0,$$
&c. &c.

$$= \int_r^{r+1} d\,\phi(\Delta) w_x = \Big[\phi(\Delta) w_x\Big]_r^{r+1} = \phi(\Delta) w_{r+1} - \phi(\Delta) w_r$$

$$= \phi(\Delta)(w_{r+1} - w_r) = \phi(\Delta)\Delta w_r$$

$$= \phi(\Delta) y_r.$$

Hence

$$\int_r^{r+1} y_x dx = \phi(\Delta) y_r.$$

Making $r=0$, we have

$$\int_0^1 y_x dx = \left\{1 + \frac{1}{2}\Delta - \frac{1}{12}\Delta^2 + \frac{1}{24}\Delta^3 - \frac{19}{720}\Delta^4 + \frac{3}{160}\Delta^5 \ldots\ldots\right\} y_0$$

$$= \frac{y_0 + y_1}{2} - \frac{1}{12}\Delta^2 y_0 + \frac{1}{24}\Delta^3 y_0 - \frac{19}{720}\Delta^4 y_0 + \frac{3}{160}\Delta^5 y_0 - \ldots\ldots .$$

Making $r=1$, we have

$$\int_1^2 y_x dx = \frac{y_1 + y_2}{2} - \frac{1}{12}\Delta^2 y_1 + \frac{1}{24}\Delta^3 y_1 - \frac{19}{720}\Delta^4 y_1 + \frac{3}{160}\Delta^5 y_1 - \ldots\ldots .$$

Making $r=2$, we have

$$\int_2^3 y_x dx = \frac{y_2 + y_3}{2} - \frac{1}{12}\Delta^2 y_2 + \frac{1}{24}\Delta^3 y_2 - \frac{19}{720}\Delta^4 y_2 + \frac{3}{160}\Delta^5 y_2 - \ldots\ldots .$$

Form similar equations for $n=3$, 4, 5,..... and finally for $r=n-1$, thus

$$\int_{n-1}^n y_x dx = \frac{y_{n-1} + y_n}{2} - \frac{1}{12}\Delta^2 y_{n-1} + \frac{1}{24}\Delta^3 y_{n-1} - \frac{19}{720}\Delta^4 y_{n-1}$$

$$+ \frac{3}{160}\Delta^5 y_{n-1} - \ldots\ldots\ldots .$$

Adding these equations together, we have the left hand member equal to

$$\int_0^n y_x dx$$

and the right hand member is

$$\left(\frac{1}{2}y_0+y_1+y_2+y_3+\ldots\ldots\ldots+y_{n-2}+\frac{1}{2}y_{n-1}\right)$$

$$-\frac{1}{12}\Delta^2\left(y_0+y_1+y_2+y_3+\ldots\ldots\ldots+y_{n-2}+y_{n-1}\right)$$

$$+\frac{1}{24}\Delta^3\left(y_0+y_1+y_2+y_3+\ldots\ldots\ldots+y_{n-2}+y_{n-1}\right)$$

$$-\frac{19}{720}\Delta^4\left(y_0+y_1+y_2+y_3+\ldots\ldots\ldots+y_{n-2}+y_{n-1}\right)$$

$$+\frac{3}{160}\Delta^5\left(y_0+y_1+y_2+y_3+\ldots\ldots\ldots+y_{n-2}+y_{n-1}\right)$$

$$-\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$$

But

$$\begin{aligned}&\Delta^m(y_0+y_1+y_2+y_3+\ldots\ldots\ldots+y_{n-2}+y_{n-1})\\&=\Delta^{m-1}(\Delta y_0+\Delta y_1+\Delta y_2+\ldots\ldots\ldots+\Delta y_{n-2}+\Delta y_{n-1})\\&=\Delta^{m-1}\left\{(y_1-y_0)+(y_2-y_1)+(y_3-y_2)+\ldots\ldots\ldots+(y_n-y_{n-1})\right\}\\&=\Delta^{m-1}(y_n-y_0).\end{aligned}$$

Hence the above equation becomes

$$\begin{aligned}\int_0^n y_x\,dx=&\frac{1}{2}y_0+y_1+y_2+y_3+\ldots\ldots\ldots+y_{n-1}+\frac{1}{2}y_n\\&-\frac{1}{12}\left(\Delta y_n-\Delta y_0\right)+\frac{1}{24}\left(\Delta^2y_n-\Delta^2y_0\right)\\&-\frac{19}{720}\left(\Delta^3y_n-\Delta^3y_0\right)+\frac{3}{160}\left(\Delta^4y_n-\Delta^4y_0\right)-\ldots\ldots\ldots.\end{aligned}$$

The differences of y_n can not be calculated from the values of y_0, y_1,.....y_n. The above result cannot therefore be employed for actual calculation. This defect can easily be remedied.

Before doing so, however, we have to establish **another interpolation formula.**

$$y_{x+n}=(1+\Delta)^n y_x=\left(\frac{1}{1+\Delta}\right)^{-n}y_x=\left(1-\frac{\Delta}{1+\Delta}\right)^{-n}y_x$$

i.e.,
$$y_{x+n}=y_x+n\left(\frac{\Delta}{1+\Delta}\right)y_x+\frac{1}{\underline{|2}}n(n+1)\left(\frac{\Delta}{1+\Delta}\right)^2 y_x$$
$$+\frac{1}{\underline{|3}}n(n+1)(n+2)\left(\frac{\Delta}{1+\Delta}\right)^3 y_x+\ldots\ldots\ldots$$
$$=y_x+n\Delta y_{x-1}+\frac{1}{\underline{|2}}n(n+1)\Delta^2 y_{x-2}$$
$$+\frac{1}{\underline{|3}}n(n+1)(n+2)\Delta^3 y_{x-3}+\ldots\ldots\ldots \quad \ldots\ldots\ldots \quad (J)$$

In this result put r for n, and then put n for $x+r$, so that
$$y_n=y_{n-r}+r\Delta y_{n-r-1}+\frac{1}{\underline{|2}}r(r+1)\Delta^2 y_{n-r-2}$$
$$+\frac{1}{\underline{|3}}r(r+1)(r+2)\Delta^3 y_{n-r-3}+\ldots\ldots\ldots.$$

Operating on this by Δ^r, we have
$$\Delta^r y_n=\Delta^r y_{n-r}+r\Delta^{r+1}y_{n-r-1}+\frac{1}{\underline{|2}}r(r+1)\Delta^{r+2}y_{n-r-2}$$
$$+\frac{1}{\underline{|3}}r(r+1)(r+2)\Delta^{r+3}y_{n-r-3}+\ldots\ldots\ldots.$$

Now make $r=1$ in this result, then
$$\Delta y_n=\Delta y_{n-1}+\Delta^2 y_{n-2}+\Delta^3 y_{n-3}+\ldots\ldots\ldots.$$

Make $r=2$, then
$$\Delta^2 y_n=\Delta^2 y_{n-2}+2\Delta^3 y_{n-3}+3\Delta^4 y_{n-4}+4\Delta^5 y_{n-5}+\ldots\ldots\ldots\ldots.$$

Similarly
$$\Delta^3 y_n=\Delta^3 y_{n-3}+3\Delta^4 y_{n-4}+6\Delta^5 y_{n-5}+10\Delta^6 y_{n-6}+\ \ldots\ldots\ldots,$$
$$\Delta^4 y_n=\Delta^4 y_{n-4}+4\Delta^5 y_{n-5}+10\Delta^6 y_{n-6}+20\Delta^7 y_{n-7}+\ldots\ldots\ldots,$$

&c. &c.

Substituting these in the expression for the integral of y_x, we obtain after simplification the following, which is **Laplace's formula for an approximate integral.**

$$\int_0^n y_x dx = \frac{1}{2}y_0 + y_1 + y_2 + y_3 + \ldots\ldots\ldots + y_{n-1} + \frac{1}{2}y_n$$

$$- \frac{1}{12}\left(\Delta y_{n-1} - \Delta y_0\right)$$

$$- \frac{1}{24}\left(\Delta^2 y_{n-2} + \Delta^2 y_0\right)$$

$$- \frac{19}{720}\left(\Delta^3 y_{n-3} - \Delta^3 y_0\right)$$

$$- \frac{3}{160}\left(\Delta^4 y_{n-4} + \Delta^4 y_0\right)$$

$$- \ldots\ldots\ldots\ldots\ldots\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (K)$$

When the successive values y_0, y_1, y_2,.....y_n do not correspond to unit difference of the independent variable x, but are those corresponding to n equidistant values of x ranging over $x_n - x_0$ and the integral is in the usual form

$$\int_{x_0}^{x_n} \phi(x) dx,$$

we have to put $x = x_0 + \frac{x_n - x_0}{n} z$, and then the above integral becomes

$$\frac{x_n - x_0}{n} \int_0^n \phi\left(x_0 + \frac{x_n - x_0}{n} z\right) dz,$$

which may be written

$$\frac{x_n - x_0}{n} \int_0^n \phi\left(x_0 + \frac{x_n - x_0}{n} x\right) dx.$$

Hence we have to multiply the formula (K) by a factor $\frac{x_n - x_0}{n}$, when the successive values of y correspond to a constant difference of x equal to that factor.

Example. As a mere illustrative example of the formula (K), let the value of $\int_{10}^{18} x^{\frac{1}{3}} dx$ be calculated approximately.

Solution :—

n	0	1	2	3	4
x	10	12	14	16	18
y	2·1544	2·2894	2·4101	2·5198	2·6207
Δy	·1350	·1207	·1097	·1009	
$\Delta^2 y$	−·0143	−·0109	−·0088		

Here

$$\frac{1}{2}y_0+y_1+y_2+\ldots\ldots+\frac{1}{2}y_n=9\cdot60685,$$

$$\frac{1}{12}\left(\Delta y_{n-1}-\Delta y_0\right)=\frac{1}{12}\left(\cdot1009-\cdot1350\right)=-\cdot00284,$$

$$\frac{1}{24}\left(\Delta^2 y_{n-2}+\Delta y_0\right)=\frac{1}{24}\left(-\cdot0088-\cdot0143\right)=-\cdot00096,$$

$$\frac{x_n-x_0}{n}=\frac{18-10}{4}=2.$$

Putting these in the formula (K) we obtain an approximate value of the integral equal to 19·2213. The value correct to the last figure is 19·2218.

Approximate Value of the Differential Coefficient of a Continuous Function in Terms of the successive Differences of the Function.

Taking the logarithms of both members of the equation (H) and expanding the right hand member, we have

$$h\frac{d}{dx}=\log(1+\Delta)$$

$$=\Delta-\frac{1}{2}\Delta^2+\frac{1}{3}\Delta^3-\frac{1}{4}\Delta^4+\ldots\ldots$$

in which the symbol Δ is now defined by

$$\Delta y_x = y_{x+h} - y_x.$$

Applying the above identical operations to any function of x, we have

$$h\frac{dy_x}{dx} = \Delta y_x - \frac{1}{2}\Delta^2 y_x + \frac{1}{3}\Delta^3 y_x - \frac{1}{4}\Delta^4 y_x + \ldots\ldots \quad \ldots\ldots \quad (L)$$

This equation enables us to calculate that value of the differential coefficient which corresponds to the first term of a given series of a function. Now we know that when an equidistant series of a function are given, the differential coefficient corresponding to the middle term, or a term near the middle, of the series is in general most accurately determined. To determine such a value of dy/dx or that corresponding to any desired value of y, we have to establish another and more general formula.

We have before obtained a well known formula of interpolation, namely,

$$y_x = y_a + n\Delta y_a + \frac{1}{\underline{|2}} n(n-1)\Delta^2 y_a + \frac{1}{\underline{|3}} n(n-1)(n-2)\Delta^3 y_a + \ldots\ldots \quad \ldots\ldots \quad (A_1)$$

in which $x = a + nh$. Differentiating the above equation with respect to n and observing that

$$\frac{dy_x}{dn} = h\frac{dy_x}{dx},$$

we have

$$h\frac{dy_x}{dx} = \Delta y_a + \frac{1}{2}(2n-1)\Delta^2 y_a + \frac{1}{6}(3n^2-6n+2)\Delta^3 y_a + \frac{1}{12}(2n-3)(n^2-3n+1)\Delta^4 y_a + \ldots\ldots \quad \ldots\ldots \quad (M)$$

This equation enables us to calculate the differential coefficient corresponding to any value of x, or of y whose value is known in terms of n. When $n=0$, the result becomes the equation (L).

Putting $n=\frac{1}{2}$, we have $x=a+\frac{1}{2}h$ and

$$h\frac{dy_x}{dx}=\Delta y_a-\frac{1}{24}\Delta^3 y_a+\frac{1}{24}\Delta^5 y_a-\ldots\ldots \quad \ldots\ldots\ldots \quad (m_1)$$

there being no term of second difference.

Putting $n=1$, we have $x=a+h$ and

$$h\frac{dy_x}{dx}=\Delta y_a+\frac{1}{2}\Delta^2 y_a-\frac{1}{6}\Delta^3 y_a+\frac{1}{12}\Delta^4 y_a-\ldots\ldots \quad \ldots \quad (m_2)$$

Putting $n=1\frac{1}{2}$, we have $x=a+1\frac{1}{2}h$ and

$$h\frac{dy_x}{dx}=\Delta y_a+\Delta^2 y_a-\frac{1}{24}\Delta^3 y_a+\ldots\ldots \quad \ldots\ldots\ldots \quad (m_3)$$

Here the term containing $\Delta^4 y_a$ does not exist.

Putting $n=2$, we have $x=a+2h$ and

$$h\frac{dy_x}{dx}=\Delta y_a+\frac{3}{2}\Delta^2 y_a+\frac{1}{3}\Delta^3 y_a-\frac{1}{12}\Delta^4 y_a+\ldots\ldots \quad \ldots\ldots \quad (m_4)$$

Example. The pressure p in lbs. per sq. foot of saturated steam corresponding to the temperature θ in Centigrade scale is as given in the third line of the following table.

n	0	1	2	3
θ	90	100	110	120
p	1463	2116	2994	4152
Δp	653	878	1158	
$\Delta^2 p$	225	280		
$\Delta^3 p$	55			

Suppose that we require the value of $\frac{dp}{d\theta}$ corresponding to the temperature $\theta=105°\,C.$

Solution :—We first form a table of differences as shown above. Since $\theta=105°$ is exactly midway between the second and third term of the series of θ, the value of n is $1\frac{1}{2}$ and we have to employ the formula (m_3), thus :—

$$h\frac{dp}{d\theta}=\Delta p_a+\Delta^2 p_a-\frac{1}{24}\Delta^3 p+\ldots\ldots$$

$$=653+225-\frac{1}{24}\times 55$$

$$=875{\cdot}7.$$

But h is here equal to 10 and therefore $\frac{dp}{d\theta}=87{\cdot}57$ for $105°\,C.$

Example. Suppose that the values of p in the third line of the following table are given and that $dp \div d\theta$ corresponding to $\theta=105°\,C.$ is required.

n	0	1	2	3	4	5	6
θ	90	95	100	105	110	115	120
p	1463	1765	2116	2524	2994	3534	4152
Δp	302	351	408	470	540	618	
$\Delta^2 p$	49	57	62	70	78		
$\Delta^3 p$	8	5	8	8			
$\Delta^4 p$	−3	+3	0				

Solution :—We form a table of differences as shown above and use the formula (m_4), omitting the column corresponding to $\theta=90°\,C.$

$$h\frac{dp}{d\theta}=\Delta p_a+\frac{3}{2}\Delta^2 p_a+\frac{1}{3}\Delta^3 p_a-\frac{1}{12}\Delta^4 p_a+\ldots\ldots\ldots$$

$$=351+\frac{3}{2}\times 57+\frac{1}{3}\times 5-\frac{1}{12}\times 3$$

$$=437{\cdot}92,$$

and therefore $$\frac{dp}{d\theta}=\frac{437{\cdot}92}{5}=87{\cdot}58.$$

We shall now deduce **another general expression for the differential coefficient of a function.** We have seen that

$$y_x=y_{a+nh}=(1+\Delta)^n y_a.$$

This can be expanded in the following form :—

$$y_x=\left(\frac{1}{1+\Delta}\right)^{-n}y_a=\left(1-\frac{\Delta}{1+\Delta}\right)^{-n}y_a$$

$$=\left\{1+n\left(\frac{\Delta}{1+\Delta}\right)+\frac{1}{\underline{2}}n(n+1)\left(\frac{\Delta}{1+\Delta}\right)^2\right.$$

$$\left.+\frac{1}{\underline{3}}n(n+1)(n+2)\left(\frac{\Delta}{1+\Delta}\right)^3+\ldots\ldots\ldots\right\}y_a$$

$$=y_a+n\Delta y_{a-h}+\frac{1}{\underline{2}}n(n+1)\Delta^2 y_{a-2h}$$

$$+\frac{1}{\underline{3}}n(n+1)(n+2)\Delta^3 y_{a-3h}+\ldots\ldots\ldots.$$

Differentiating this result with respect to n, we obtain

$$h\frac{dy_x}{dx}=\Delta y_{a-h}+\frac{1}{2}(2n+1)\Delta^2 y_{a-2h}+\frac{1}{6}(3n^2+6n+2)\Delta^3 y_{a-3h}$$

$$+\frac{1}{12}(2n+3)(n^2+3n+1)\Delta^4 y_{a-4h}+\ldots\ldots\quad\ldots\ (N)$$

in which $x=a+nh$.

When $n=0$, x becomes a and we obtain

$$h\frac{dy_a}{dx}=\Delta y_{a-h}+\frac{1}{2}\Delta^2 y_{a-2h}+\frac{1}{3}\Delta^3 y_{a-3h}+\ldots\ldots\ldots.\quad\ldots\ldots\ldots\quad(n_1)$$

When $n=-1$, we have $x=a-h$ and

$$h\frac{dy_{a-h}}{dx}=\Delta y_{a-h}-\frac{1}{2}\Delta^2 y_{a-2h}-\frac{1}{6}\Delta^3 y_{a-3h}$$

$$-\frac{1}{12}\Delta^4 y_{a-4h}-\ldots\ldots\ldots \quad \ldots\ldots\ldots\ldots\ldots \quad (n_2)$$

When $n=-2$, we have $x=a-2h$ and

$$h\frac{dy_{a-2h}}{dx}=\Delta y_{a-h}-\frac{3}{2}\Delta^2 y_{a-2h}+\frac{1}{3}\Delta^3 y_{a-3h}$$

$$+\frac{1}{12}\Delta^4 y_{a-4h}+\ldots\ldots\ldots \quad \ldots\ldots\ldots\ldots\ldots \quad (n_3)$$

In the equation (m_2) put $a+h$ for a and in the equation (n_2) put $a+3h$ for a, and adding the resulting equations together, we obtain

$$h\frac{dy_{a+2h}}{dx}=\frac{1}{2}\left(\Delta y_{a+h}+\Delta y_{a+2h}\right)-\frac{1}{12}\left(\Delta^3 y_a+\Delta^3 y_{a+h}\right)$$

$$-\frac{1}{24}\left(\Delta^4 y_{a-h}-\Delta^4 y_{a+h}\right)-\ldots\ldots\ldots \quad \ldots\ldots\ldots\ldots\ldots \quad (n_4)$$

This equation enables us to calculate the differential coefficient of y corresponding to the third term y_{a+2h} of a given series y_a, y_{a+h}, y_{a+2h}, y_{a+3h}, y_{a+4h} consisting of five terms, neglecting the fourth and higher differences.

Again in the equation (n_3) put $a+4h$ for a and add the resulting equation to the equation (m_4), we then obtain

$$h\frac{dy_{a+2h}}{dx}=\frac{1}{2}\left(\Delta y_a+\Delta y_{a+3h}\right)+\frac{3}{4}\left(\Delta^2 y_a-\Delta^2 y_{a+2h}\right)$$

$$+\frac{1}{6}\left(\Delta^3 y_a+\Delta^3 y_{a+h}\right)+\ldots\ldots\ldots \quad \ldots\ldots\ldots\ldots\ldots \quad (n_5)$$

in which the term containing the fourth differences does not exist. This is a good form of a formula for calculating the

differential coefficient corresponding to the third term of a given series consisting of five terms.

In the equation (L) put $a+3h$ for x and in the equation (n_1) put $a+3h$ for a, and adding the resulting equations together, we obtain

$$h\frac{dy_{a+3h}}{dx}=\frac{1}{2}\left(\Delta y_{a+2h}+\Delta y_{a+3h}\right)+\frac{1}{4}\left(\Delta^2 y_{a+h}-\Delta^2 y_{a+3h}\right)$$

$$+\frac{1}{6}\left(\Delta^3 y_a+\Delta^3 y_{a+3h}\right)+\ldots\ldots\ldots\ldots\ldots\ldots \quad (n_6)$$

by which we can calculate the differential coefficient of the middle term of a series consisting of seven terms. Many other formulæ may be obtained easily.

Working out the same example before given, by the equation (n_4), we have

$$h\frac{dp_{105}}{d\theta}=\frac{1}{2}\left(408+470\right)-\frac{1}{12}\left(5+8\right)=437{\cdot}93,$$

therefore $$\frac{dp_{105}}{d\theta}=\frac{1}{5}\times 437{\cdot}93=87{\cdot}59.$$

If we use the equation (n_5) we have

$$h\frac{dp_{105}}{d\theta}=\frac{1}{2}\left(351+540\right)+\frac{3}{4}\left(57-70\right)+\frac{1}{6}\left(5+8\right)$$

$$=437{\cdot}92,$$

therefore $$\frac{dp_{105}}{d\theta}=\frac{1}{5}\times 437{\cdot}92=87{\cdot}58.$$

If we use the equation (n_6) we have

$$h\frac{dp_{105}}{d\theta}=\frac{1}{2}\left(408+470\right)+\frac{1}{4}\left(57-70\right)+\frac{1}{6}\left(8+8\right)$$

$$=438{\cdot}42,$$

therefore $\frac{dp_{105}}{d\theta} = \frac{1}{5} \times 438{\cdot}42 = 87{\cdot}68.$

This result is probably the most accurate.

We give the following numerical examples which only serve to show the degree of accuracy that can be attained by the use of some of the foregoing formulæ.

In the subjoined table the numbers in the line headed y are the natural logarithms of the numbers in the line headed x. The problem is to calculate the value of dy/dx corresponding to $x=1{\cdot}4$. Of course there is no real use of this calculation, because it is well known that the differential coefficient of the natural logarithm of any number is the reciprocal of that number.

n	0	1	2	3	4	5	6
x	1·1	1·2	1·3	1·4	1·5	1·6	1·7
y	·0953	·1823	·2624	·3365	·4055	·4700	·5306
Δy	·0870	·0801	·0741	·0690	·0645	·0606	
$\Delta^2 y$	−·0069	−·0060	−·0051	−·0045	−·0039		
$\Delta^3 y$	·0009	·0009	·0006	·0006			

(1) Given the natural logarithms of 1·2, 1·3, 1·4, 1·5, and 1·6; required the value of

$$d(\log_e x)/dx \qquad \text{for} \qquad x=1{\cdot}4.$$

Using the formula (m_4), we have

$$\cdot1\frac{dy}{dx} = \cdot0801 + \frac{3}{2}\left(-\cdot0060\right) + \frac{1}{3}\left(\cdot0009\right)$$

$$= \cdot07140,$$

therefore $\frac{dy}{dx} = \cdot 7140.$

Using the formula (n_4), we have

$$\cdot 1\frac{dy}{dx} = \frac{1}{2}\left(\cdot 0741 + \cdot 0690\right) - \frac{1}{12}\left(\cdot 0006 + \cdot 0006\right)$$

$$= \cdot 07145,$$

therefore $\frac{dy}{dx} = \cdot 7145.$

Using the formula (n_5), we have

$$\cdot 1\frac{dy}{dx} = \frac{1}{2}\left(\cdot 0801 + \cdot 0645\right) + \frac{3}{4}\left(-\cdot 0060 + \cdot 0045\right) + \frac{1}{6}\left(\cdot 0009 + \cdot 0006\right)$$

$$= \cdot 071425,$$

therefore $\frac{dy}{dx} = \cdot 71425$

(2) Given the natural logarithms of 1·1, 1·2, 1·3, 1·4, 1·5, 1·6, and 1·7; required the value of

$$d(\log_e x)/dx \quad \text{for} \quad x = 1\cdot 4.$$

Using the formula (n_6), we have

$$\cdot 1\frac{dy}{dx} = \frac{1}{2}\left(\cdot 0741 + \cdot 0690\right) + \frac{1}{4}\left(-\cdot 0060 + \cdot 0045\right) + \frac{1}{6}\left(\cdot 0009 + \cdot 0006\right)$$

$$= \cdot 071425,$$

therefore $\frac{dy}{dx} = \cdot 71425.$

The value of $\frac{dy}{dx}$ correct to five decimal places, for $x = 1\cdot 4$ is $\frac{1}{1\cdot 4} = \cdot 71429.$

To develop $\Delta^n y_x$ in a Series proceeding by the Differential Coefficients of y_x.

From the formula (H) we obtain at once

$$\Delta^n = \left(e^{h\frac{d}{dx}} - 1\right)^n$$

in which the symbol Δ is defined by

$$\Delta y_x = y_{x+h} - y_x.$$

The above can be expanded as follows:—

$$\Delta^n = h^n\left(\frac{d}{dx}\right)^n + \frac{n}{2}h^{n+1}\left(\frac{d}{dx}\right)^{n+1} + \frac{n(3n+1)}{24}h^{n+2}\left(\frac{d}{dx}\right)^{n+2}$$

$$+ \frac{n^2(n+1)}{48}h^{n+3}\left(\frac{d}{dx}\right)^{n+3} + \ldots\ldots$$

Hence we have

$$\Delta^n y_x = h^n\frac{d^n y_x}{dx^n} + \frac{n}{2}h^{n+1}\frac{d^{n+1}y_x}{dx^{n+1}} + \frac{n(3+1)}{24}h^{n+2}\frac{d^{n+2}y_x}{dx^{n+2}}$$

$$+ \frac{n^2(n+1)}{48}h^{n+3}\frac{d^{n+3}y_x}{dx^{n+3}} + \ldots\ldots \quad \ldots\ldots \quad (P)$$

When $n=1$, we have

$$\Delta y_x = h\frac{dy_x}{dx} + \frac{1}{2}h^2\frac{d^2y_x}{dx^2} + \frac{1}{6}h^3\frac{d^3y_x}{dx^3} + \frac{1}{24}h^4\frac{d^4y_x}{dx^4} + \ldots\ldots \quad (p_1)$$

When $n=2$, we have

$$\Delta^2 y_x = h^2\frac{d^2y_x}{dx^2} + h^3\frac{d^3y_x}{dx^3} + \frac{7}{12}h^4\frac{d^4y_x}{dx^4} + \frac{1}{4}h^5\frac{d^5y_x}{dx^5} + \ldots\ldots \quad (p_2)$$

When $n=3$, we have

$$\Delta^3 y_x = h^3\frac{d^3y_x}{dx^3} + \frac{3}{2}h^4\frac{d^4y_x}{dx^4} + \frac{5}{4}h^5\frac{d^5y_x}{dx^5} + \frac{3}{4}h^6\frac{d^6y_x}{dx^6} + \ldots\ldots \quad (p_3)$$

In the preparation of the reprint of Mr. Barlow's Tables of Squares, Cubes, Square Roots, Cube Roots, and Reciprocals by

Prof. A. De Morgan, certain formulæ were employed in checking the second differences of square roots, cube roots and reciprocals. These formulæ can be obtained from equation (p_2). Thus in the case of a square root, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{-\frac{1}{2}},$$

$$\frac{d^2y}{dx^2} = \frac{d^2}{dx^2}\left(x^{\frac{1}{2}}\right) = -\frac{1}{4}x^{-\frac{3}{2}},$$

$$\frac{d^3y}{dx^3} = \frac{d^3}{dx^3}\left(x^{\frac{1}{2}}\right) = \frac{3}{8}x^{-\frac{5}{2}}.$$

Making $h=1$ and taking first two terms of the expansion (p_2), we have

$$\Delta^2\sqrt{x} = -\frac{1}{4}x^{-\frac{3}{2}} + \frac{3}{8}x^{-\frac{5}{2}}$$

$$= -\frac{1}{4}x^{-\frac{3}{2}}\left(1 - \frac{3}{2}x^{-1}\right)$$

$$= -\frac{1}{4}x^{-\frac{3}{2}}\left(1 + x^{-1}\right)^{-\frac{3}{2}} \qquad \text{nearly}$$

$$= -\frac{1}{4(x+1)^{\frac{3}{2}}}.$$

This was used for checking the second differences of square roots of numbers greater than 1250. Similarly in the case of cube root and reciprocal, we easily obtain

$$\Delta^2\sqrt[3]{x} = -\frac{2}{9}\sqrt[3]{\frac{1}{x(x+1)^3}} \qquad \text{and} \qquad \Delta^2\frac{1}{x} = \frac{2}{(x+1)^3}.$$

To develop Σy_x in a Series proceeding by the Differential Coefficients of y_x.

Let $$\Sigma y_x = y_a + y_{a+1} + y_{a+2} + y_{a+3} + \ldots\ldots\ldots + y_{x-1}.$$

Taking the difference, we obtain

$$\Delta \Sigma y_x = y_x$$

Hence

$$\Sigma y_x = \Delta^{-1} y_x.$$

Strictly speaking, this should be written in the form

$$\Sigma y_x = \Delta^{-1} y_x + \text{constant}.$$

Because the first term in the given series might have been y_{a+r} instead of y_a where r is any integer, positive or negative, and still the same result would have been arrived at. The case is exactly analogous to adding a constant to an indefinite integral. Now we had before

$$1 + \Delta = e^{\frac{d}{dx}},$$

from which

$$\Delta^{-1} = \Sigma = \frac{1}{e^{\frac{d}{dx}} - 1}$$

$$= \left(\frac{d}{dx}\right)^{-1} \times \frac{\frac{d}{dx}}{e^{\frac{d}{dx}} - 1}.$$

This last fraction can be expanded in the following form (see Todhunter's Dif. Cal., page 95 and also Williamson's Dif. Cal., page 93):—

$$\frac{\frac{d}{dx}}{e^{\frac{d}{dx}} - 1} = 1 - \frac{1}{2}\frac{d}{dx} + \frac{B_1}{\underline{|2}}\frac{d^2}{dx^2} - \frac{B_3}{\underline{|4}}\frac{d^4}{dx^4} + \frac{B_5}{\underline{|6}}\frac{d^6}{dx^6} - \ldots\ldots$$

where Bernoulli's numbers are

$$B_1 = \frac{1}{6}, \quad B_3 = \frac{1}{30}, \quad B_5 = \frac{1}{42}, \quad B_7 = \frac{1}{30},$$

$$B_9 = \frac{5}{66}, \quad B_{11} = \frac{691}{2730}, \quad B_{13} = \frac{7}{6},$$

$$B_{15} = \frac{3617}{510}, \quad B_{17} = \frac{43867}{798}, \quad B_{19} = \frac{1222277}{2310}.$$

Hence
$$\Sigma y_x = \left\{\frac{d}{dx}\right\}^{-1} \times \left\{\text{the series just obtained}\right\} y_x$$
$$= \int \left\{\text{the series just obtained}\right\} y_x dx$$
$$= \int \left\{ y_x - \frac{1}{2}\frac{dy_x}{dx} + \frac{B_1}{\underline{|2}}\frac{d^2}{dx^2} y_x - \ldots\ldots \right\} dx.$$

Therefore
$$\Sigma y_x = C + \int y_x dx - \frac{1}{2} y_x + \frac{1}{12}\frac{dy_x}{dx} - \frac{1}{720}\frac{d^3 y_x}{dx^3} + \frac{1}{30240}\frac{d^5 y_x}{dx^5} - \ldots\ldots \quad (M)$$

which is Euler-Maclaurin formula.

Example. Required to find the sum
$$1^n + 2^n + 3^n + \ldots\ldots + x^n.$$

Let S be the sum so that $S = x^n + \Sigma x^n$, where $y_x = x^n$.

Now
$$\int y_x dx = \frac{x^{n+1}}{n+1}, \qquad \frac{dy_x}{dx} = n x^{n-1},$$
$$\frac{d^3 y_x}{dx^3} = n(n-1)(n-2)x^{n-3}, \text{ \&c.}$$

Hence
$$S = x^n + \Sigma x^n$$
$$= x^n + C + \frac{x^{n+1}}{n+1} - \frac{1}{2}x^n + \frac{1}{12} n x^{n-1}$$
$$- \frac{1}{720} n(n-1)(n-2)x^{n-3}$$
$$+ \frac{1}{30240} n(n-1)(n-2)(n-3)(n-4)x^{n-5} - \ldots\ldots.$$

The constant C can be determined by applying a particular value of the variable x. Thus making $x = 1$,
$$0 = C + \frac{1}{n+1} - \frac{1}{2} + \frac{n}{12} - \frac{1}{720} n(n-1)(n-2) + \ldots\ldots$$

and putting $n=1, 2, 3, 4, 5$, &c., we find that the constant C is zero, for all even values of n,

$$C=-\frac{1}{12} \text{ for } n=1, \qquad C=-\frac{1}{120} \text{ for } n=3,$$

$$C=-\frac{1}{252} \text{ for } n=5, \qquad \text{&c.}$$

Example. To find the sum $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots+\frac{1}{x^2}$.

Let the sum be $S=\frac{1}{x^2}+\Sigma\frac{1}{x^2}$.

$$\int y_x dx=\int\frac{dx}{x^2}=-\frac{1}{x}, \qquad \frac{dy_x}{dx}=-\frac{2}{x^3},$$

$$\frac{d^2y_x}{dx^2}=\frac{2\cdot3}{x^4}, \qquad \frac{d^3y_x}{dx^3}=-\frac{2\cdot3\cdot4}{x^5}, \qquad \text{&c.}$$

Therefore
$$S=\frac{1}{x^2}+C-\frac{1}{x}-\frac{1}{2}\frac{1}{x^2}-\frac{1}{12}\frac{2}{x^3}+\frac{1}{720}\cdot\frac{2\cdot3\cdot4}{x^5}+\ldots\ldots$$

$$=C-\frac{1}{x}+\frac{1}{2x^2}-\frac{1}{6x^3}+\frac{1}{30x^5}-\ldots\ldots$$

But $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots$ ad. inf. is known to be $\frac{1}{6}\pi^2$. Hence the contant $C=\frac{1}{6}\pi^2$.

Example. To find the sum

$$1^2+(1^2+2^2)+(1^2+2^2+3^2)+\ldots\ldots+(1^2+2^2+3^2+\ldots\ldots+x^2).$$

Let y_x be the required sum, then

$$\Delta y_x=1^2+2^2+3^2+\ldots\ldots+(x+1)^2$$

$$\Delta^2y_x=(x+2)^2$$

$$\Delta^3y_x=2x+5,$$

$$\Delta^4y_x=2,$$

and all the higher differences are zero.

Therefore $y_0=0, \quad \Delta y_0=1, \quad \Delta^2 y_0=4, \quad \Delta^3 y_0=5,$

$\Delta^4 y_0=2.$

Hence formula (A) gives

$$y_x=0+x+\frac{1}{2}x(x-1)4+\frac{1}{6}x(x-1)(x-2)5$$

$$+\frac{1}{24}x(x-1)(x-2)(x-3)2$$

$$=\frac{1}{12}x(x+1)^2(x+2).$$

Example. To find the sum

$$1\cdot2\cdot3+2\cdot3\cdot4+3\cdot4\cdot5+4\cdot5\cdot6+\ldots\ldots+n(n+1)(n+2).$$

Result :—The sum of the series is

$$S=\frac{1}{4}(n)(n+1)(n+2)(n+3).$$

The sum of a series of reciprocals of factorial expression can be expressed in a finite number of terms.

Example. $$S=\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\ldots\ldots+\frac{1}{x(x+1)}$$

$$=\left\{\frac{1}{1}-\frac{1}{2}\right\}+\left\{\frac{1}{2}-\frac{1}{3}\right\}+\left\{\frac{1}{3}-\frac{1}{4}\right\}+\ldots\ldots$$

$$+\left\{\frac{1}{x-1}-\frac{1}{x}\right\}+\left\{\frac{1}{x}-\frac{1}{x+1}\right\}$$

$$=1-\frac{1}{x+1}=\frac{x}{x+1}.$$

Example. $$\frac{1}{1\cdot3\cdot5}+\frac{1}{3\cdot5\cdot7}+\frac{1}{5\cdot7\cdot9}+\ldots\ldots+\frac{1}{x(x+2)(x+4)}$$

$$=\frac{1}{4}\left\{\frac{1}{1\cdot3}-\frac{1}{3\cdot5}\right\}+\frac{1}{4}\left\{\frac{1}{3\cdot5}-\frac{1}{5\cdot7}\right\}+\ldots\ldots$$

$$+\frac{1}{4}\left\{\frac{1}{x(x+2)}-\frac{1}{(x+2)(x+4)}\right\}$$

$$= \frac{1}{4}\left\{\frac{1}{1\cdot 3} - \frac{1}{(x+2)(x+4)}\right\}$$

$$= \frac{1}{4}\cdot\frac{(x+1)(x+5)}{(x+2)(x+4)}.$$

Tokyo, Aug. 1900.

XI.

Theory of Fly Wheel as Applied to Turbines and High Speed Engines.

[*The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. IV, No. 5, June,* 1901.]

The use of a fly wheel as applied to a steam or gas engine is usually stated to be the equalization or smoothing up of the periodic fluctuation of speed due to periodic excess or deficiency of energy given to and received from the prime mover. Many formulæ that have been given for the weight and size of a fly wheel are based upon what has been stated above and can therefore be derived from

$$\frac{1}{2} I \omega^2 = \frac{k}{q} E \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (A)$$

in which the left hand member is the well known expression for the kinetic energy of the fly wheel with reference to the axis of rotaion, k the coefficient of periodic fluctuation of energy, q the coefficient of fluctuation of speed, and E the mean amount of energy exerted during one period. In the case of an ordinary single cylinder engine, E is the work done per stroke by the engine at its normal speed.

Now the function of the governor and of the regulator for the supply of the working fluid to the prime mover (such as a throttle valve or an expansion gear of a steam engine, a cylindrical sluice gate or a set of movable guide vanes of a turbine, &c.) is to make the mean supply of energy *during any time* equal to the mean work of resistance overcome during the same time.

In high speed steam engines, the period in which the regular fluctuation of energy, and consequently of speed, occurs is very small indeed, being $\frac{1}{5}$, or $\frac{1}{10}$, or $\frac{1}{15}$ of a second, or even still less. In a turbine or a Pelton wheel working against a uniform resistance, there is no such periodic fluctuation, the supply of energy being practically perfectly constant. Suppose that in a high speed engine or a turbine, the load is suddenly withdrawn in part or in whole. The speed then increases to a certain extent usually very small; the governor then moves from its normal position and causes the working fluid regulator to act upon the steam or the water, either directly or indirectly. From the instant of the withdrawal of the load to the beginning of action of the working fluid regulator, there must be some interval of time, though it may be a second or two, or a small fraction of a second. Let this time be t_1. Again let the time measured from the beginning of action of the working fluid regulator to the end of its action be t_2. In high speed steam engines and gas engines, t_2 is very small; but in turbines it is usually considerable. Thus in the big Fourneyron turbines at Niagara, t_2 from full power to no power is 12 seconds; in Mr. M. A. Replogle's mechanical differential relay governor, it is said that the ratio of gearing is such as to make t_2 from full power to no power equal to 15 to 25 seconds.

Now with regard to engines in which t_1+t_2 is greater than the time of one revolution, if we have a proper fly wheel to guard against the variation of speed due to sudden removal of load, the engine will certainly take better care of itself against the periodic variation of speed, and it is the coefficient of *this* variation, q, which the formula (A) deals with. In turbines there does not exist at all the quantity q; but owing to the nature of the machine it is very difficult to make t_2 and therefore t_1+t_2 so small as in high speed steam engines. The use of a proper fly wheel for turbines to guard against a sudden variation of load is therefore even more important than in the case of steam engines.

To apply to these cases the formula (A) or others that can be derived therefrom is evidently wrong. In the following pages the writer has investigated the matter from first principles and obtained some useful formulæ. The results of this theory of fly wheel are applicable directly to turbines and Pelton wheels. These can be applied also to steam engines when the following statement holds good:—

Suppose that the periodic variation of speed in a high speed engine has been made sufficiently small to be of no practical consequence, then the twisting moment on the crank shaft due to the steam pressure, though varying to a large amount during each revolution, is in reality a rapid succession of equal couples applied during equal intervals; and so far as the effect extending over a large number of such intervals is concerned, the twisting moment may be regarded as uniform, when the resistance overcome is uniform.

Let

I=the moment of inertia of a fly wheel and all other masses

rotating with it; for shortness we will hereafter speak of a "fly wheel" simply,

F_0R=a constant twisting moment applied to the fly wheel shaft, the magnitude of the force being F_0, acting at a radius R from the axis of the shaft,

ω_0=the constant angular velocity of the fly wheel, when the moment of resistance overcome is equal to F_0R,

ω=the variable angular velocity at any time t,

θ=the angle in radians over which the fly wheel is turned corresponding to time t; we measure θ from the instant at which the time t is zero.

Suppose that a certain portion of the entire load denoted by cF_0 where c is a fraction less than unity, has been suddenly withdrawn. The moment of unbalanced force acting on the fly wheel is then cF_0R and from Dynamics we have

$$I\frac{d\omega}{dt}=cF_0R \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

the right hand side of which is constant from the instant of the sudden withdrawal of the load till the beginning of action of the working fluid regulator. Hence integrating and noting that $\omega=\omega_0$ for $t=0$, we have

$$I(\omega-\omega_0)=cF_0Rt. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

The equation (1) can also be integrated in the form of kinetic energy, thus:—

$$\frac{1}{2}I(\omega^2-\omega_0^2)=cF_0R\theta. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

Denoting by H the horse power transmitted by the shaft of the fly wheel at the normal speed ω_0 and taking foot, pound, second, and radian as our units, we have

$$H=\frac{F_{\circ}R\omega_{\circ}}{550}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Eliminating $F_{\circ}R$ between (2) and (4), we have

$$I(\omega-\omega_{\circ})=c\frac{550\,H}{\omega_{\circ}}t,$$

from which we obtain

$$\frac{\omega-\omega_{\circ}}{\omega_{\circ}}=\frac{c}{2}\cdot\frac{550\,H}{\frac{1}{2}I\omega_{\circ}^{2}}\cdot t.$$

Let $t_{\circ}$ be the time in which the fly wheel starting from rest can attain the normal speed $\omega_{\circ}$ under the action of the constant twisting moment $F_{\circ}R$, there being, in the mean time, no external resistance during that interval. This quantity $t_{\circ}$ is perfectly definite for all existing turbines, steam engines, gas engines, &c., and can be calculated approximately or else can be determined by experiments without much difficulty. The dynamical equation of motion during the time $t_{\circ}$ is the same as the equation (1), except that c is now unity, thus:—

$$I\frac{d\omega}{dt}=F_{\circ}R.$$

Integrating and remembering that $\omega=0$ for $t=0$ and $\omega=\omega_{\circ}$ for $t=t_{\circ}$, we obtain

$$I\omega_{\circ}=F_{\circ}Rt_{\circ}.$$

Combining this with equation (4), we obtain

$$t_{\circ}=2.\frac{\frac{1}{2}I\omega_{\circ}^{2}}{550\,H}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

Hence the expression for the coefficient of variation of angular velocity in a time t becomes

$$\frac{\omega-\omega_{\circ}}{\omega_{\circ}}=\frac{ct}{t_{\circ}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

If in this result we put c equal to unity, there is obtained the case of a sudden removal of the entire load, as will happen with a turbine driving an electric dynamo machine, which is in turn driving a single train of tramway cars. The coefficient of speed variation at the end of the time t_1 (which as has been mentioned before, is the time measured from the instant of the removal of the load to the beginning of action of the working fluid regulator) is expressed by $ct_1 \div t_0$. The importance of making t_0 large and t_1 small is thus seen.

From the end of the time t_1, the working fluid regulator comes into action during a time t_2 and in consequence the twisting moment, which can be denoted by FR, is now a variable. But the dynamical equation of motion remains in the same form, namely,

$$I\frac{d\omega}{dt} = FR.$$

As the variation of F is not known, the only possible way to solve the above equation is to make a probable assumption as regards the action of the working fluid regulator. A most simple and plausible assumption is the following:—

$$F = F_0\left(c - \frac{t}{\tau}\right), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

in which τ is the time required by the working fluid regulator to complete its action from full power to no power. When the entire load has been withdrawn suddenly, c is unity and the time at the end of which the force entirely ceases to act is τ. When any fraction c of the entire load has been withdrawn, the corresponding time of action is $c\tau$, which has been denoted by t_2. Substituting the assumed value of F in the dynamical equation, we have

$$I\frac{d\omega}{dt}=F_{\circ}R\left(c-\frac{t}{\tau}\right). \quad \ldots\ldots\ldots\ldots\ldots \quad (8)$$

Integrating and putting $\omega=\omega_1$ for $t=0$, we obtain

$$I(\omega-\omega_1)=F_{\circ}Rt\left(c-\frac{t}{2\tau}\right). \quad \ldots\ldots\ldots\ldots \quad (9)$$

Observe that ω_1 is the value of ω obtained from equation (6) by putting $t=t_1$. Eliminating $F_{\circ}R$ by means of equations (4) and (5), we obtain

$$\frac{\omega-\omega_1}{\omega_{\circ}}=\frac{1}{t_{\circ}}t\left(c-\frac{t}{2\tau}\right). \quad \ldots\ldots\ldots\ldots \quad (10)$$

This expression for the coefficient of variation of speed becomes a maximum for $t=c\tau$, that is, it becomes a maximum at the end of action of the working fluid regulator. Putting ω_2 for the corresponding maximum value of ω, we have

$$\frac{\omega_2-\omega_1}{\omega_{\circ}}=\frac{c^2}{2}\frac{\tau}{t_{\circ}}. \quad \ldots\ldots\ldots\ldots \quad (11)$$

Adding the equations (6) and (11), and observing that $c.\tau=t_2$, we obtain the coefficient of the greatest variation of speed due to sudden withdrawal of load :—

$$\frac{\omega_2-\omega_{\circ}}{\omega_{\circ}}=\frac{c}{t_{\circ}}(t_1+\tfrac{1}{2}c\tau)$$

$$=\frac{c}{t_{\circ}}(t_1+\tfrac{1}{2}t_2). \quad \ldots\ldots\ldots\ldots \quad (12)$$

This equation enables us to determine the ability of a given fly wheel acting, in conjunction with the governor and the working fluid regulator, as an equalizer of speed. However excellent the governor and other regulators may be, if the $t_{\circ}$ of the fly wheel be not large enough, it would be impossible to attain to a sufficient degree of uniformity of speed. It is to be observed that

in no case can the quantity c be absolutely equal to unity; because even when the external load is entirely removed, there still remains the friction of the mechanism of the prime mover acting as a resisting load, so that when the supply of the water to a turbine or of the steam to a steam engine is completely shut off, the force denoted by F is actually negative.

Let us now try to find the value of t_0 for prime movers used in actual practice. In the big Fourneyron turbines at Niagara the total weight of the revolving part which serves as a fly wheel is 152,000 lbs., of which 79,000 lbs. are the weight of the field ring of the alternator. What the diameter of this field ring is, we do not know; however, if 9 ft. be asumed as the effective diameter of gyration, we shall probably be not far from truth. The turbine shaft, which is a steel tube, is 38″ in diameter and for this we assume 30″ as the effective diameter of gyration. Each turbine works under an average head of 136 ft., discharges 430 cub. ft. of water per second, runs at 250 revolutions per minute, and develops 5000 horse power at about 75 per cent. efficiency. With these data the total kinetic energy of the fly wheel is found to be 18,250,000 ft. lbs. and therefore from equation (5), we have

$$t_0 = 2 \times \frac{18{,}350{,}000}{550 \times 5000} = 13{\cdot}3 \text{ seconds.}$$

This estimate is probably too low; the true value is perhaps 15 seconds or more.

From practice of the Westinghouse single acting steam engines used for electric lighting, Mr. F. M. Rites (Trans. A. S. M. E., XIV, 100) derived the following rule:—The energy in ft. lbs. of fly wheel per horse power should not be less than

$$\frac{850{,}000}{n};$$

in which n is the number of revolutions per minute. This rule is easily obtained from the equation (A) by putting $k \div q$ equal to a particular value and expressing E in terms of horse power and the number of revolutions per minute. The above rule of Mr. Rites may be written in the form

$$t_0 = 2.\frac{\frac{1}{2}I\omega_0^2}{550H} = \frac{3090}{n}.$$

If we assume $n = 360$ as an average value, we obtain

$$t_0 = \frac{3090}{360} = 8{\cdot}6 \text{ seconds.}$$

We take another example. A Pelton wheel 31″ in external diameter works under a head 430 ft. at a speed of 620 revolutions per minute, developing from 200 to 250 horse power. The fly wheel on the shaft is 4 ft. in diameter and weighs 4000 lbs. Neglecting the inertia of the Pelton wheel, but taking the external diameter of the fly wheel as the diameter of gyration, we find by calculation that $t_0 = 19$ and $12{\cdot}2$ seconds corresponding to 200 and 250 horse power respectively. This Pelton wheel is regulated by means of a deflecting nozzle and a Replogle electric relay governor. Supposing that the action of this governor is instantaneous, that is, taking $t_1 = 0$, and that it takes $3\frac{1}{2}$ seconds to deflect the nozzle completely and also if we assume 10 per cent. frictional resistance, the variation of speed upon the removal of the entire load as expressed by equation (12) becomes

$$\frac{\omega_2 - \omega_0}{\omega_0} = \frac{1}{2}c^2\frac{\tau}{t_0} = \frac{1}{2} \times {\cdot}9^2 \times \frac{3\frac{1}{2}}{19} = {\cdot}075,$$

that is, $7\frac{1}{2}$ per cent.

In the case of the big Fourneyron turbines at Niagara, also neglecting t_1, we have

$$\frac{\omega_2-\omega_\circ}{\omega_\circ}=\frac{1}{2}c^2\frac{\tau}{t_\circ},$$

in which $\tau=12$ seconds and $t_\circ$ may be taken at 15 seconds, so that allowing 5 per cent. as the greatest permissible variation of speed, we obtain $c=\cdot37$, that is to say, with our assumed data and condition, the greatest load that can be removed suddenly with safety is 37 per cent. of the full load. This is about 1850 horse power.

It sometimes happens that a turbine or a Pelton wheel has to work under a head different from what it was originally intended to work under. Let us see what change of the value of the I of fly wheel is necessary under such a circumstance. Let h be the head of water in ft. and Q the discharge in cubic ft. per second, then the horse power of the motor is

$$H=const.\ Qh$$

and

$$Q=Av,$$

where A is the effective sectional area of the passage at any part of the hydraulic motor and v the velocity of water across that section. Now this v and the velocity of the wheel itself at any particular part of it bear a constant ratio to each other and both can be expressed by $const.\ \sqrt{2gh}$. Hence for one and the same wheel we have

$$\begin{aligned} H&=const.\ h^{\frac{3}{2}}\\ &=const.\ v^3\\ &=const.\ \omega_\circ^3. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (13)\end{aligned}$$

Putting this value of horse power in equation (5), we obtain

$$t_0 = \frac{const.\ I}{\omega_0} = \frac{const.\ I}{\sqrt{h}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

From equation (12) we see that in order that the turbine may be equally steady under a different head, t_0 must remain the same and therefore we must have

$$I = const.\ \omega_0 = const.\ \sqrt{h}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (15)$$

which amounts to saying that to make one and the same turbine equally steady when worked under different heads of water, the fly wheel on the shaft should have its moment of inertia proportioned to the square root of the head or to the number of revolutions per minute. If for example, I_0 be the value of I for a particular turbine suitable when working under a head of 180 ft., the same turbine, if employed under a head of 430 ft., ought to have the I increased to $I_0\sqrt{430 \div 180} = 1{\cdot}55 I_0$ in order that it may have the same degree of steadiness with the same governing gear.

It may be here remarked that a steam engine is often run at a speed different from that originally intended, without altering the pressure and other conditions. In this case it is evident that

$$H = const.\ \omega_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (13')$$

And then reasoning as in the above case of a turbine, we obtain

$$I = \frac{const.}{\omega_0}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (15')$$

which compare with equation (15). Hence running a steam engine at an increased speed without any other change, means increased steadiness, both as regards sudden variations of loads and the periodic fluctuation of energy during successive revolutions. The last clause follows from the fact that the equation

(A) leads to an expression for the I equal to a constant divided by the *square* of the angular velocity.

The reasoning and the results that have been given thus far, have a special reference to sudden removal of the load on the prime mover. Change the signs of F_0 in the equations (1), (2), (3), (7), and (8); and also change the signs of the left hand members of the equations (6), (10), (11), and (12). We have then a similar theory relating to the case of sudden application of the load on the prime mover.

Let us conclude these pages by saying that complaints of defective speed regulation are frequent in many water power plants and that in the writer's opinion the simplest plan that can be deviced toward the attainment of a uniform speed is to have fly wheels of sufficiently huge moment of inertia.

XII.

Torsion Test of Cast-Iron.

By

A. Inokuty, *Kogakuhakushi, Kogakushi*

and

Fuji Tanaka, *Kogakushi.*

[*Report of College of Engineering, Tokyo Imperial University, No. V, pp. 7–9, June,* 1903.]

XIII.

Determination of Coefficients of Discharge from a Rectangular Notch.

By

A. Inokuty, *Kogakuhakushi, Kogakushi*

and

S. Utimaru, *Kogakushi.*

[*Report of College of Engineering, Tokyo Imperial University, No. V, pp. 4–6, June,* 1903.]

XIV.

Result of Tests of Vacuum Gauges.

By

A. Inokuty, *Kogakuhakushi, Kogakushi*

and

S. Utimaru, *Kogakushi.*

[*Report of College of Engineering, Tokyo Imperial University, No. V, pp.* 1–3, *June*, 1903.]

XV.

Strength of Flat Part of a Steam Boiler Fitted with a Doubling Plate.

[*The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. VI, No. 9, June,* 1903.]

Suppose that a flat part of a steam boiler, of thickness t_1 and bearing steam pressure of p lbs. per sq. inch, is fitted with a doubling plate of thickness t_2 with the object of strengthening the main plate. Lloyd's Rule for flat plates places a limit to the smallest value of the thickness t_2 of the doubling plate equal

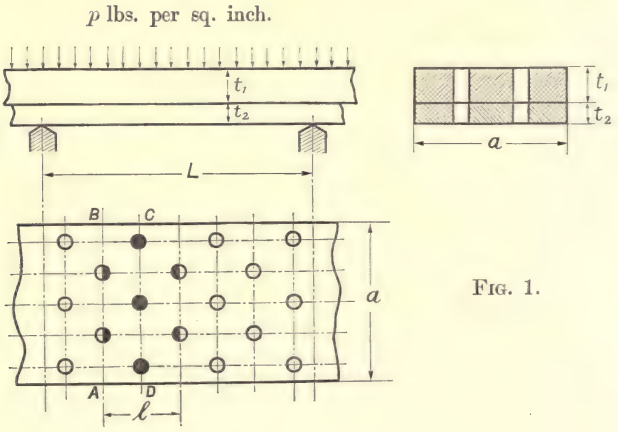


FIG. 1.

to $\frac{2}{3}t_1$ and estimates the strength of the compound plates as equivalent to that of a solid plate having a thickness $t_1+\frac{1}{2}t_2$, so that the formula to be used is

$$p=C\times\frac{(t_1+\frac{1}{2}t_2)^2}{(pitch)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

in which C is a constant having different values according to circumstances.

Now if the riveting of the two plates be perfectly tight so that the two plates can have no slipping action between them, then the steam pressure can be increased to

$$p=C\times\frac{(t_1+t_2)^2}{(pitch)^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

the intensity of bending stress allowed remaining the same. Such a perfect joint cannot, of course, be expected in practice.

If, on the other hand, the riveting be so very bad that the two plates behave as two independent plates simply bent to the same curvature, then with the same intensity of bending stress, the pressure of steam can be increased to

$$p=C\times\frac{t_1^2}{(pitch)^2}\times\left\{1+\left(\frac{t_2}{t_1}\right)^3\right\}. \quad \ldots\ldots\ldots \quad (3)$$

This result can be arrived at in the following manner. The amount of deflection of a beam is

$$\delta=\text{const.}\times\frac{WL^3}{EI},$$

where $W=$ total load on the beam $=paL$, $I=\frac{1}{12}at^3$, and $E=$ the modulus of direct elasticity of the material. The above may be written thus

$$\delta=\text{const.}\times\frac{p}{t^3},$$

for all beams of the same material, of the same span and of the same mode of loading. Now since the two plates are bent to the same curvature, the deflections are of course the same, so that if we suppose p_1 and p_2 to be the two parts into which the steam pressure p can be divided as being borne by the two plates respectively, we have

$$\frac{p_1}{t_1^3}=\frac{p_2}{t_2^3}.$$

But since

$$p_1+p_2=p,$$

we have

$$p_1=\frac{t_1^3}{t_1^3+t_2^3}p \quad \text{and} \quad p_2=\frac{t_2^3}{t_1^3+t_2^3}p.$$

Hence t_1 being supposed to be greater than t_2, the allowable steam pressure p for the compound plate is obtained by substituting the above value of p_1 in the formula

$$p_1=C\times\frac{t_1^2}{(pitch)^2},$$

the result being the formula (3).

Supposing that the riveting of the two plates is perfectly tight so that the compound plate can behave as one solid plate, let us consider the shearing stress produced in the sections of the rivets. This stress is expressed by

$$iq=\frac{FG}{zI},$$

where q is the intensity of the said shearing stress in the rivet sections formed by the touching faces of the two plates; i the ratio of the area of the rivet sections to the area of the touching faces; and the other letters have the usual meanings attached to them. Referring to Fig. 1, we see that the value of the fraction i is

$$i=\frac{5A}{al},$$

where A is the sectional area of each rivet. The geometrical moment of the area of the section of the doubling plate is

$$G=\int_{\frac{1}{2}(t_1-t_2)}^{\frac{1}{2}(t_1+t_2)} yady=\Big[\tfrac{1}{2}ay^2\Big]_{\frac{1}{2}(t_1-t_2)}^{\frac{1}{2}(t_1+t_2)}$$

$$=\tfrac{1}{2}at_1t_2;$$

the geometerical moment of inertia of the whole section of the compound plate is

$$I=\tfrac{1}{12}a(t_1+t_2)^3;$$

and the width of the section z of the beam is constant and is equal to a. The above general formula for the shearing stress therefore becomes

$$iq=\frac{6Ft_1t_2}{a(t_1+t_2)^3}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

For a beam loaded uniformly all over, whether it be supported at the ends or fixed at the ends, the shearing force F at a distance x from either end is

$$F=\frac{W}{L}(\tfrac{1}{2}L-x),$$

where W is the total load on the beam and is equal to paL, p being the intensity of uniform pressure on the beam per unit area. We can therefore write F thus:—

$$F=pa(\tfrac{1}{2}L-x).$$

Substituting this in the equation (4), we obtain

$$iq=\frac{6p(\frac{1}{2}L-x)t_1t_2}{(t_1+t_2)^3}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4')$$

Now for a given pressure p and a given thickness t_1 of the main plate, the expression (4) or (4′) for the intensity of shearing stress q becomes maximum when

$$t_2=\tfrac{1}{2}t_1.$$

Putting this value of t_2 in (4′) and also making $x=0$, we obtain

$$iq=\frac{4}{9}\frac{pL}{t_1},$$

which gives the greatest possible value of the shearing stress q in the rivets, supposing that the doubling plate is of half the thickness of the main plate and that the riveting is perfectly tight so that there cannot be any slipping action between the two plates. It will be found that with ordinary proportions of doubling plate, the value of the stress q calculated by the above formula (4′) is rather too high, showing that the method of fixing a doubling plate is not well suited to the purpose in view, that is to say, that it is a poor method to adopt for withstanding a bending action. Here is an actual example of a doubling plate in the boiler of Spanish armoured cruiser "Infanta Maria Teresa" (ENGINEERING, June 22, 1894.) End plate $\frac{7}{8}''$ thick, doubling plate $\frac{13}{16}''$ thick, space between two nests of tubes measured between the nearest vertical centre lines of two sets of tubes 14″, diameter of rivets 1″, pitch of rivets $3\frac{1}{2}''$, steam pressure 150 lbs. per sq. inch. Here we have the ratio of the area of the rivet sections to the area of the touching faces of the two plates equal to

$$i=\frac{\frac{\pi}{4}\times(1)^2}{(3\frac{1}{2})^2}=0.0641,$$

and the shearing stress produced in the section of the rivets equal to

$$q = \frac{6 \times 150 \times \frac{1}{2} \times 14 \times \frac{7}{8} \times \frac{13}{16}}{0.0641 \times (\frac{7}{8} + \frac{13}{16})^3}$$

$$= 14{,}500 \text{ lbs per sq. inch.}$$

Now if the riveted joints be perfect, the shearing stress in the rivets is thus seen to be very high, and therfore it will not be safe to employ the formula (2) alone; if the riveting be imperfect, it is not rational to employ (2). The formula (3) is true only in the case of an extremely bad riveted joint; it is useful only in finding the limit of pressure p to which we are quite safe to raise it, irrespective of the size and number of rivets.

The distribution of normal stresses upon a transverse section of the compound plate considerd as a beam is shown in the accompanying Fig. 2. It is here assumed that the two plates have a slight amount of slipping action along the touching faces. The stresses in each plate may be considered to consist of two parts, one being the bending stress and the other longitudinal uniform compressive or tensile stress. The first of these are represented by f_1 and f_2, and the second by f_c and f_t. Putting M for the bending moment at any section, the moment of stresses resisting the bending action becomes

$$M = \tfrac{1}{6} f_1 a\, t_1^2 + \tfrac{1}{6} f_2 a\, t_2^2 + \tfrac{1}{2}(t_1 + t_2) f_t a\, t_2 \quad \ldots\ldots \quad (5)$$

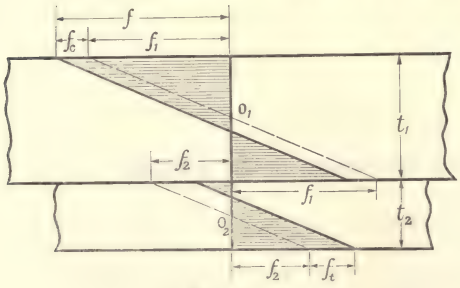


Fig. 2.

Since the total compressive stress in the main plate must be equal to the total tensile stress in the doubling plate, we have

$$f_t t_2 = f_c t_1. \quad \ldots\ldots \quad (6)$$

The two plates are supposed to touch each other after bending and therefore they have the same curvature so that

$$\frac{f_1}{f_2}=\frac{t_1}{t_2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

Now in the case of a perfectly tight riveted joint in which no slipping action can take place

$$f_t=f_1 \quad \text{and} \quad f_c=f_2;$$

while in the case of a perfectly loose joint,

$$f_t=0 \quad \text{and} \quad f_c=0.$$

The actual values of f_t and f_c may be anything between these limits.

1° Suppose we take $f_t=\frac{1}{2}f_1$ and $f_c=\frac{1}{2}f_2$. Eliminating f_t by this assumption and f_2 by (7), from equation (5), we obtain

$$M=\tfrac{1}{6}f_1 a t_1^2+\tfrac{1}{6}f_1 a\frac{t_2^3}{t_1}+\tfrac{1}{4}(t_1+t_2)a f_1 t_2$$

$$=\tfrac{1}{6}f_1 a\frac{(t_1+t_2)}{t_1}(t_1^2+\tfrac{1}{2}t_1 t_2+t_2^2). \quad \ldots\ldots\ldots \quad (8)$$

The greatest stress at the top of the main plate is

$$f=f_1+f_c$$
$$=f_1+\tfrac{1}{2}f_2$$
$$=f_1+\tfrac{1}{2}\frac{t_2}{t_1}f_1$$
$$=\left(1+\tfrac{1}{2}\frac{t_2}{t_1}\right)f_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

and the greatest stress at the bottom of the doubling plate is

$$f_2+f_t=\frac{t_2}{t_1}f_1+\tfrac{1}{2}f_1$$
$$=\left(\frac{t_2}{t_1}+\tfrac{1}{2}\right)f_1, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

which is less than f so long as t_2 is less than t_1. When t_2 is equal to t_1 both expressions for the stresses become the same, agreeably to what we expect. Eliminating f_1 from (8) by means of (9) and writing m for $\frac{t_2}{t_1}$ for shortness, we obtain

$$M=\tfrac{1}{6} f a t_1^2 \times \frac{(1+m)(1+\frac{1}{2}m+m^2)}{(1+\frac{1}{2}m)}. \quad \ldots\ldots \quad (11)$$

2° Suppose we take $f_t=\frac{1}{3}f_1$ and $f_c=\frac{1}{3}f_2$, then proceeding in the same manner as in the last case, we obtain

$$M=\tfrac{1}{6} f a t_1^2 \times \frac{(1+m)(1+m^2)}{1+\frac{1}{2}m}. \quad \ldots\ldots\ldots \quad (12)$$

These equations can be written in a form suitable for calculating the working pressure p of the steam, thus:—

$$p=C \times \frac{t_1^2}{(pitch)^2} \times \frac{(1+m)(1+\frac{1}{2}m+m^2)}{(1+\frac{1}{2}m)}. \quad \ldots\ldots \quad (11')$$

$$p=C \times \frac{t_1^2}{(pitch)^2} \times \frac{(1+m)(1+m^2)}{(1+\frac{1}{2}m)}. \quad \ldots\ldots\ldots \quad (12')$$

For all values of t_2 less than 0·626 times t_1, the working pressure p calculated by (12′) is less than that calculated by Lloyd's rule (1). For greater values of t_2, the formula (12′) gives a slightly greater pressure.

[TABLE.

Table *Showing the Values of the Coefficient k in the Formula :—*

$$Working\ Pressure = k \times C \times \frac{(thickness\ of\ main\ plate)^2}{(pitch\ of\ rivets)^2}$$

The ratio of the thickness of doubling plate to that of main plate, $m=\frac{t_2}{t_1}$	Coefficient k in the formula :—				
	(2) Perfect joint in which $f_t=f_1$, $f_c=f_2$.	(11′) Joint in which $f_t=\frac{1}{2}f_1$, $f_c=\frac{1}{2}f_2$.	(12′) Joint in which $f_t=\frac{1}{3}f_1$, $f_c=\frac{1}{3}f_2$.	(1) Lloyd's Rule	(3) Most imperfect joint in which $f_t=0$, $f_c=0$.
	$k=(1+m)^2$	$k=\frac{(1+m)(1+\frac{1}{2}m+m^2)}{(1+\frac{1}{2}m)}$	$k=\frac{(1+m)(1+m^2)}{(1+\frac{1}{2}m)}$	$k=(1+\frac{1}{2}m)^2$	$k=(1+m)^3$
0	1	1	1	1	1
$\frac{1}{4}$	1·56	1·32	1·18	1·27	1·016
$\frac{1}{2}$	2·25	1·80	1·50	1·56	1·125
·626	2·64	2·11	1·72	1·72	1·245
$\frac{3}{4}$	3·06	2·47	1·99	1·89	1·422
1	4·00	3·33	2·67	2·25	2·000

Since Lloyd's rule puts a condition that t_2 should not be less than $\frac{3}{8}t_1$, we see from the above result of calculation that using the formula (1) is equivalent to assuming f_t slightly less than $\frac{1}{3}f_1$ and f_c slightly less than $\frac{1}{3}f_2$. The writer considers that *the formula* (12′) *can be safely employed for all thicknesses* t_2 *of the doubling plate.*

XVI.

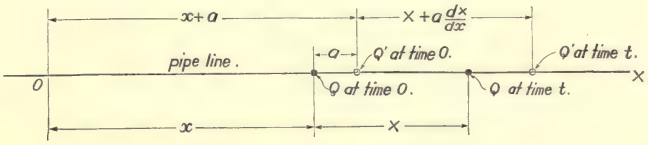
Sudden Stoppage of Water in a Pipe of Uniform Section.

[*The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. VII, No.* 11, *September*, 1904.]

When the flow of water in a pipe of uniform section is suddenly arrested, an extra pressure above the normal statical pressure is produced and the intensity of this extra pressure depends on the velocity of flow and also on the elasticity of the pipe and of water itself. In the following pages, a process of calculation of this pressure and of the velocity of propagation of a pulsation through water in the pipe is given. The problem may be conveniently divided into two cases, namely, first the case in which the pipe is supposed to be infinitely rigid and water alone elastic and secondly the case in which both the pipe and water are supposed elastic.

First Case. The pipe is supposed rigid and water alone elastic.

It is the pressure p above the normal statical pressure that is considered in this problem and accordingly the effects are independent of whether the pipe is vertical or horizontal.



Let p=the pressure due to the stoppage of flow, being that above the normal statical pressure,

w=the weight of water per cubic unit,

v=the velocity of water at time t,

K=the modulus of cubical elasticity of water,

x=the distance of a particle Q measured along the pipe in the direction of motion, from a fixed section O of the pipe, at time 0,

X=the distance travelled by the particle Q in the time t,

$x+a$=the distance of another particle Q' measured along the pipe from the same section O at time 0.

The distance travelled by the particle Q' in the time t may then be expressed by

$$X+a\frac{dX}{dx}$$

and the distance between the two particles Q and Q' which was a at time 0, is now

$$a+a\frac{dx}{dt}$$

at time t. A portion of water lying between two sections Q and Q' and which occupied Aa cubic feet has increased by $Aa\frac{dX}{dx}$ cubic feet at time t. Hence the cubical strain is $\frac{dX}{dx}$ and consequently

$$p=-K\frac{dX}{dx}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

In the above, A is the sectional area of the pipe. Now the velocity is $v=\frac{dX}{dt}$ and the force causing the acceleration $\frac{d^2X}{dt^2}$ in the mass $\frac{w}{g}Adx$ is $-Adp$ and hence

$$-Adp=\frac{w}{g}Adx\frac{d^2X}{dt^2},$$

therefore

$$-\frac{dp}{dx}=\frac{w}{g}\frac{d^2X}{dt^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Differentiating (1), we have

$$-\frac{dp}{dx}=K\frac{d^2X}{dx^2}.$$

Hence by (2) we have

$$c^2\frac{d^2X}{dx^2}=\frac{d^2X}{dt^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

in which

$$c=\sqrt{\frac{gK}{w}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Differentiating (3) with regard to t and writing v for $\frac{dX}{dt}$, we have

$$c^2\frac{d^2v}{dx^2}=\frac{d^2v}{dt^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

which is an equation of a well known type occurring in several physical problems. The complete solution of (5) is

$$v=f\left(t-\frac{x}{c}\right)+\phi\left(t+\frac{x}{c}\right), \quad \ldots\ldots\ldots\ldots \quad (6)$$

where f and ϕ are arbitrary functions. Consider first the motion represented by the first term alone, on the right hand side of this equation. Since $f\left(t-\frac{x}{c}\right)$ is unaltered when t and x are increased by τ and $c\tau$, respectively, it is plain that any particular state of motion which existed at the point x at time t has been

transferred at time $t+\tau$ to the point $x+c\tau$. Hence the said state of motion advances unchanged with a constant velocity c. In other words, we have a certain state of motion travelling with a constant velocity c in the direction of x positive. In the same way the second term on the right hand side of (6) represents a certain state of motion travelling with velocity c in the direction of x negative.

Now let our problem be as follows. Suppose that water is flowing through a pipe OX from X toward O with a uniform velocity v_0 and that a rigid diaphragm having no thickness is suddenly introduced at the section $x=0$, that is, at O. Required to find the ensuing state of motion. Consider the pipe only for positive values of x.

The velocity of water is $v=-v_0$ at every section of the pipe at time $t=0$; and at the section O where $x=0$, v must be such a function of the time t that v was $-v_0$ till $t=0$, and ever afterwards v there is 0. The equation (6) then becomes

$$v=-v_0+f\left(t-\frac{x}{c}\right), \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

in which f is such that $f(t)$ is 0 till $t=0$, and then becomes v_0 for all values of t. The state of motion is therefore as follows:— Until the time $t=\frac{x}{c}$, the velocity is $-v_0$ at any place; and after $t=\frac{x}{c}$, the velocity is 0. Since $v=\frac{dX}{dt}$, integrating (7) with respect to the time t, we have

$$X=-v_0t+F\left(t-\frac{x}{c}\right), \quad \ldots\ldots\ldots\ldots\ldots \quad (8)$$

where F is such that $\frac{d}{dy}F(y)=f(y)$, whatever y may be. Hence from (1) we obtain

$$p = -K\frac{d}{dx}\left\{-v_0 t + F\left(t-\frac{x}{c}\right)\right\}$$

$$= \frac{K}{c} f\left(t-\frac{x}{c}\right)$$

$$= \sqrt{\frac{wK}{g}} f\left(t-\frac{x}{c}\right).$$

This shows that at any section the pressure p is zero till the time $t=\frac{x}{c}$ and then becomes

$$p = v_0\sqrt{\frac{wK}{g}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

and remains of this value. This is the state of pressure produced at every section of the pipe, the subsidence of the motion of water and the generation of pressure happenning at the same time. This state of thing is propagated through the water in the pipe with a constant velocity c given by the equation (4).

Second Case. The water in the pipe as well as the material of the pipe itself is considered elastic.

Let r=the internal radius of the pipe,

E=the modulus of direct elasticity of the material of the pipe,

T=the thickness of the pipe,

f=the circumferential tensile stress in the pipe due to the pressure p.

By the usual formula for the strength of a pipe under an internal fluid pressure, we have

$$f = \frac{pr}{T}.$$

The amount of elongation of the circumference due to this tensile stress is

$$\delta c = \frac{f 2\pi r}{E}$$

and therefore the increase of the radius of the pipe is

$$\delta r = \frac{fr}{E}.$$

The increase of the sectional area of the pipe is

$$\begin{aligned}\delta A &= \pi(r+\delta r)^2 - \pi r^2 \\ &= 2\pi r \delta r \\ &= \frac{2\pi r^2 f}{E} \\ &= \frac{2\pi r^3 p}{ET}.\end{aligned}$$

This multiplied by the distance a between the two particles Q and Q' gives the increase of the contents of the pipe for that length. Dividing this increase of contents by the original area of the pipe, we obtain the amount of the relative motion of Q and Q' due to the distention of the pipe, viz :—

$$\frac{2rpa}{ET}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

The volume of water Aa between the two sections at Q and Q' is diminished by the pressure p by the amount

$$\frac{pAa}{K}.$$

This divided by the area of the pipe gives the amount of the relative motion of Q and Q' due to the compression of water, viz :—

$$\frac{pa}{K}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

Hence the total relative motion of the two particles Q and Q'

in the time t is the sum of (10) and (11). But it is evident that this motion is also represented by $a\frac{dX}{dx}$.

Hence $$a\frac{dX}{dx} = -\frac{2rpa}{ET} - \frac{pa}{K},$$

from which we have

$$p = -K\left(\frac{ET}{2rK+ET}\right)\frac{dX}{dx}. \quad \ldots\ldots\ldots\ldots \quad (12)$$

This corresponds to the equation (1) of the First Case. The remaining part of the investigation will be exactly the same as before. Hence the uniform velocity with which the subsidence of motion and the generation of pressure are propagated through water is given by

$$c' = \sqrt{\frac{gK}{w}\cdot\frac{ET}{2rK+ET}}. \quad \ldots\ldots\ldots\ldots \quad (13)$$

At any section the pressure is zero till the time $t=\frac{x}{c'}$ and then becomes

$$p = v_0\sqrt{\frac{wK}{g}\cdot\frac{ET}{2rK+ET}},$$

and remains of this value. If we assume that there is no distention of the pipe, the velocity with which the state of motion is propagated is given by (4), being the velocity of sound in water. The modulus of cubical elasticity of water is K=294,000 lbs. per sq. inch and

$$c = \sqrt{\frac{gK}{w}} = \sqrt{\frac{32{\cdot}15\times 144\times 294{,}000}{62{\cdot}3}}$$

$$= 4674, \text{ say, } 4670 \text{ ft. per second,}$$

which is usually taken at 4700 ft. per sec.

Example. Water is flowing through a pipe 5 ft. in dia., $\frac{1}{4}''$ thick, 12,800 ft. long, made of steel plates, supplying water to a turbine at the rate of 6 ft. per second. If the flow of water is stopped suddenly, what will be the pressure produced and what will be the velocity with which this pressure is propagated through the water in the pipe?

The E for steel may be taken at 30,000,000 lbs. per sq. inch and K for water at 294,000 lbs. per sq. inch. The velocity of propagation is then

$$c' = \sqrt{\frac{gK}{w} \cdot \frac{ET}{2rK+ET}}$$

$$= \sqrt{\frac{32{\cdot}15 \times 144 \times 294{,}000}{62{\cdot}3} \cdot \frac{144 \times 30{,}000{,}000 \times \frac{1}{48}}{2 \times 2{\cdot}5 \times 144 \times 294{,}000 + 144 \times 30{,}000{,}000 \times \frac{1}{48}}}$$

$$= 4674 \times 0{\cdot}5462 = 2553, \text{ say, } 2550 \text{ ft. per sec.}$$

The pressure produced above the normal statical pressure is

$$p = v_0 \sqrt{\frac{wK}{g} \cdot \frac{ET}{2rK+ET}}$$

$$= v_0 \times 9058 \times 0{\cdot}5462 = 49476 v_0 \text{ lbs. per sq. ft.}$$

$$= 34{\cdot}36 v_0 \text{ lbs. per sq. inch}$$

$$= 34{\cdot}36 \times 6 = 206 \text{ lbs. per sq. inch.}$$

When the material of the tube containing water has a low value of elastic constant, such as india-rubber, the velocity of propagation c' through water becomes very low. Thus in an india-rubber tube $\frac{5}{16}''$ internal dia. and $\frac{1}{8}''$ thick, the value of c' calculated by (13) is only 79 ft. per second. The E for india-rubber is here assumed to be 155 lbs. per sq. inch, being the value of the constant found by Dr. Winkler for low tensile stress up to about half kilogramme per sq. centimetre of initial sectional area.

XVII.

Sizes of Ready Made Engineering Articles.

[*The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. VII, No.* 11, *September,* 1904.]

Engines and machines, that are ready made and kept in stock for immediate sale at any time, are usually made in a series of definite sizes and numbered 1, 2, 3, &c. These numbers are sometimes fractional or zero; the use of these fractional numbers or zero arises from the fact that there sometimes occur frequent demands for a size smaller than the smallest in a given series of sizes or for a size intermediate between two consecutive sizes.

Fixing the sizes of a machine or an engine for ready sale is therefore of some importance. Looking through catalogues of engineering articles by different manufacturers, it is found that in many cases the sizes of an article proceed quite irregularly and in other cases quite regularly. A few actual examples of the sizes of ready made articles are given below. Horizontal selfcontained short-stroke engines manufactured by Messrs. Ruston and Proctor as given in the makers' list are as follows:—

Table I.

Engine No.	1	2	3	4	5	6	7	8
Average IP	7	11	13	17	20	24	30	36
Δ IP	4	2	4	3	4	6	6	
Δ^2IP	−2	+2	−1	+1	+2	0		

The numbers in the third line of Table I are formed by taking the differences of the average horse powers given in the second line; and the numbers in the fourth line of the table are the differences of the numbers in the third line. These second differences being rather irregular, it appears that in the makers' list above referred to, the horse power of the No. 2 engine may better be 9 or 10 instead of 11 and the following sizes are probably slightly better.

Table II.

Engine No.	1	2	3	4	5	6	7	8
Average IP	7	9	12	16	20	25	30	36
Δ IP	2	3	4	4	5	5	6	
Δ^2IP	1	1	0	1	0	1		

From a list of particulars of vertical short-stroke engines manufacutured by Messrs. Marshall we have the following:—

Table III.

Engine No.	1	2	3	4	5	6	7	8	9	10
Average IP	5	$7\frac{1}{2}$	$10\frac{1}{2}$	$13\frac{3}{4}$	16	20	25	$30\frac{1}{2}$	36	42
Δ IP	$2\frac{1}{2}$	3	$3\frac{1}{4}$	$2\frac{1}{4}$	4	5	$5\frac{1}{2}$	$5\frac{1}{2}$	6	
Δ^2IP	$\frac{1}{2}$	$\frac{1}{4}$	-1	$1\frac{3}{4}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$		

These numbers for the horse power proceed quite regularly except those for No. 5 and No. 6. We give the following as a slightly better series of horse powers.

Table IV.

Engine No.	1	2	3	4	5	6	7	8	9	10
Average IP	5	$7\frac{1}{4}$	10	$13\frac{1}{4}$	17	21	$25\frac{1}{2}$	$30\frac{1}{2}$	36	42
Δ IP	$2\frac{1}{4}$	$2\frac{3}{4}$	$3\frac{1}{4}$	$3\frac{3}{4}$	4	$4\frac{1}{2}$	5	$5\frac{1}{2}$	6	
Δ^2IP	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$		

The horse powers of tandem compound engines manufactured by E. R. and F. Turner are as follows:—

Table V.

Engine No.	1	2	3	4	5	6	7	8
Average IP	44	55	68	82	96	110	124	140
Δ IP	11	13	14	14	14	14	16	
Δ^2IP	2	1	0	0	0	2		

This series is quite regular, but the following is perhaps slightly better.

Table VI.

Engine No.	1	2	3	4	5	6	7	8
Average IP	44	55	67	80	94	109	125	142
Δ IP	11	12	13	14	15	16	17	
Δ^2IP	1	1	1	1	1	1		

The following sizes of pipes and the delivery of water by centrifugal pumps are quoted from the 1891 Catalogue of Steam Pumping and Hydraulic Machinery manufactured by Messrs. Tangyes Limited of Birmingham.

Table VII.

Dia. of pipe in inches d	2	3	4	5	6	7	8	9	10	11	12
Delivery in gals. per min. G	40	100	200	350	520	750	1000	1300	1600		2300
Velocity of water in ft. per min. v	294	327	368	412	425	450	460	472	460		470

The successive differences of G are somewhat irregular as shown below.

Table VIII.

Dia. d	2	3	4	5	6	7	8	9	10
Deliv. G	40	100	200	350	520	750	1000	1300	1600
ΔG	60	100	150	170	230	250	300	300	
$\Delta^2 G$	40	50	20	60	20	50	0		
$\Delta^3 G$	+10	−30	+40	−40	+30	−50			

The following values of the deliveries G are slightly better.

Table IX.

Dia. d	2	3	4	5	6	7	8	9	10
G	40	100	200	340	520	735	1000	1280	1600
ΔG	60	100	140	180	215	245	280	320	
$\Delta^2 G$	40	40	40	35	30	35	40		
$\Delta^3 G$	0	0	−5	−5	+5	+5			

When the delivery by a pump is given it is the velocity of water through the pipe that determines the proper diameter of the pipe. Hence in this case a suitable method of finding the sizes of the pumps would be as follows: First obtain a good series of numbers representing the deliveries as in Table IX and secondly let the velocities of water corresponding to the several deliveries be determined in accordance with some rule, which may be such as will make the velocity increase slowly with the increase of delivery. Simple calculations then give the required diameters of the pipes, which may be numbered 1, 2, &c.

From a certain catalogue of machines we find the following sizes of ratchet screw jacks:—

Table X.

Size No.	1	2	3	4	5	6	7	8
Test capacity W tons	5	6	8	10	12	16	20	30
ΔW	1	2	2	2	4	4	10	
$\Delta^2 W$	1	0	0	2	0	6		

Here the difference of capacities between No. 4 and No. 5 is rather too small and that between No. 7 and No. 8 is too large. The following series is, we think, better than the above.

Table XI.

No.	1	2	3	4	5	6	7	8
W	5	6	8	10	13	17	23	30
ΔW	1	2	2	3	4	6	7	
$\Delta^2 W$	1	0	1	1	2	1		

We consider that the following series of sizes of screw jacks up to 30 tons test load is still better.

Table XII.

No.	1	2	3	4	5	6	7	8
W	3	5	$7\frac{1}{2}$	11	$14\frac{1}{2}$	19	$24\frac{1}{2}$	30
ΔW	2	$2\frac{1}{2}$	$3\frac{1}{2}$	$3\frac{1}{2}$	$4\frac{1}{2}$	$5\frac{1}{2}$	$5\frac{1}{2}$	
$\Delta^2 W$	$\frac{1}{2}$	1	0	1	1	0		

Or else the next series.

Table XIII.

No.	1	2	3	4	5	6	7	8
W	3	5	8	11	15	19	24	30
ΔW	2	3	3	4	4	5	6	
$\Delta^2 W$	1	0	1	0	1	1		

In another catalogue we find the following sizes of hydraulic lifting jacks, in which W is the test load in tons.

Table XIV.

No.	1	2	3	4	5	6	7	8	9	10	11	12
W	3	4	6	8	10	12	15	20	30	40	50	60
ΔW	1	2	2	2	2	3	5	10	10	10	10	
$\Delta^2 W$	1	0	0	0	1	2	5	0	0	0		

The following series of sizes is better.

Table XV.

No.	1	2	3	4	5	6	7	8	9	10	11	12
W	3	5	8	12	16	21	26	32	38	45	52	60
ΔW	2	3	4	4	5	5	6	6	7	7	8	
$\Delta^2 W$	1	1	0	1	0	1	0	1	0	1		

In choosing suitable sizes and numbers for an article, in the first place it is necessary to settle the smallest and the largest sizes that are likely to be within the range of ready sale. Secondly the number of numbers must be settled; and thirdly the differences of sizes of successive numbers should be so distributed that the degree of inconvenience sustained by purchasers in selecting nearest sizes for their requirements, should be the least possible. Probably no exact solutions of these questions are ever possible, judgment and experience alone enabling us to settle the question only approximately. The foregoing actual examples and the following process of forming the successive sizes of an article may however be of assistance to engineers and merchants who deal with realy made articles.

Suppose that it is required to manufacture a large number of steam engines of a certain type for sale, the smallest being 5 horse power and the largest 47 horse power. It is assumed that eight sizes are sufficient to meet all requirements within the proposed range of power. Now the total difference between the smallest and the largest size is $47-5=42$ horse power and this 42 must be distributed among $(8-1)$ differences. The average difference is therefore 6. Now for the successive differences we have

$$6+\phi(x),$$

in which $\phi(x)$ may be any function of x, and x represents the order of sizes, that is, the number; the only condition which ϕ must satisfy is

$$\Sigma_1^{n-1}\phi(x)=0,$$

where n is the total number of sizes.

The simplest possible form which the function ϕ can assume is zero. In this case the horse powers of the engines proceed by equal differences, being 5, 11, 17, 23, 29, 35, 41, 47

The next simplest form of ϕ is

$$\phi(x)=c(x-\tfrac{1}{2}n),$$

where c is a constant. If we make $c=\cdot7$, we have

$$6+\phi(x)=6+\cdot7(x-\tfrac{1}{2}\times 8)=3\cdot2+\cdot7x$$

and the successive differences are 3·9, 4·6, 5·3, 6·0, 6·7, 7·4, 8·1. Adding to 5 these differences successively, we obtain the following series of horse powers; 5, 8·9, 13·5, 18·8, 24·8, 31·5, 38·9, 47·0. Another series of sizes is obtained by making the constant c equal to 1; thus:—

Table XVI.

No.	1	2	3	4	5	6	7	8
$\mathbb{P}$	5	8	12	17	23	30	38	47
$\Delta\mathbb{P}$	3	4	5	6	7	8	9	
$\Delta^2\mathbb{P}$	1	1	1	1	1	1		

The form of the function ϕ next in order of simplicity is a quadratic expression; thus

$$\phi(x) = Ax^2 + Bx + C,$$

in which A, B, and C are constants. The summation of a series consisting of $(n-1)$ terms of a quadratic expression is

$$\begin{aligned}\Sigma_1^{n-1}\phi(x) &= \Sigma_1^{n-1}(Ax^2 + Bx + C)\\ &= \tfrac{1}{6}(n-1)n(2n-1)A + \tfrac{1}{2}(n-1)nB + (n-1)C.\end{aligned}$$

This result ought to be zero in accordance with the condition previously given, and any one of the three constants A, B, and C may be expressed in terms of the other two; thus

$$C = -\tfrac{1}{6}n(2n-1)A - \tfrac{1}{2}nB.$$

This being substituted in the expression for $\phi(x)$, we obtain

$$\phi(x) = A\{x^2 - \tfrac{1}{6}n(2n-1)\} + B(x - \tfrac{1}{2}n).$$

In the example now being considered,

$$\Delta\mathbb{P} = 6 + \phi(x),$$

therefore

$$\begin{aligned}\Delta^2\mathbb{P} &= 6 + \phi(x+1) - \{6 + \phi(x)\}\\ &= \phi(x+1) - \phi(x)\\ &= \Delta\phi(x)\\ &= 2Ax + B + A,\end{aligned}$$

and

$$\Delta^3\mathbb{P} = 2A.$$

Let $\Delta \mathbb{P} = 2{\cdot}5$ and $\Delta^2 \mathbb{P} = 1{\cdot}3$ for $x=1$, then $A=-0{\cdot}04$ and $B=1{\cdot}42$, from which the following series of horse powers are obtained :—

Table XVII.

No.	1	2	3	4	5	6	7	8
$\mathbb{P}$	5·00	7·50	11·30	16·32	22·48	29·70	37·90	47·00
$\Delta \mathbb{P}$	2·50	3·80	5·02	6·16	7·22	8·20	9·10	
$\Delta^2 \mathbb{P}$	1·30	1·22	1·14	1·06	·98	·90		
$\Delta^3 \mathbb{P}$	−·08	−·08	−·08	−·08	−·08			

The size of an article corresponding to any number x may be put in the form :—

$$\text{Size of No. } 1 + \Sigma_1^{x-1}\{\text{average difference} + \phi(x)\}.$$

In the same example suppose that the best horse power of No. 5 engine is 21. Let it be required to find the horse powers of other numbers. In this case the size of No. $1=5$, $x=5$, the average difference $=6$, and $\phi(x)$ may be assumed equal to $c(x-\frac{1}{2}n) = c(x-4)$. Also the size of No. 5 is required to be 21 horse power. Hence we have

$$21 = 5 + \Sigma_1^{4}\{6 + c(x-4)\},$$

from which the constant c is found equal to $\frac{4}{3}$. The series of horse powers required then becomes 5, 7, $10\frac{1}{3}$, 15, 21, $28\frac{1}{3}$, 37, 47.

Although, as has been stated before, no exact solution of the question in hand would ever be possible, it is thought that the process of forming the successive differences of numbers in a series of sizes and of adjusting the numbers by means of differences, give facilities towards the solution of the question.

XVIII.

Shearing Test of Timbers.

By

A. Inokuty, *Kogakuhakushi, Kogakushi*

and

Fuji Tanaka, *Kogakushi.*

[*The Journal of the College of Engineering, Tokyo Imperial University, Japan, Vol. II, No. 3, January*, 1905.]

Two important conditions for an accurate shearing test are, first, that the direction of the load on the test piece be in or parallel to the plane over which the shearing takes place and, secondly, that there should be no possibility of bending stress being produced. For materials like wrought iron and steel, the second of the above conditions can be attained by accurately shaping the test piece and fitting it in a shackle arranged like a knuckle joint, or by making a punching test as in an ordinary punching machine, or by holding the test piece rigidly by means of screw bolts at a short distance apart and applying the shearing force between the bolts. None of these methods being well suited for the case of a shearing test of timber along or across the grain and on a plane parallel to the direction of the fibre, we adopted the following very simple and ready method of testing.

Each test piece was prepared to the form and dimensions shown in Fig. 1, the fibres of the wood running in the direction of the height of the test piece, *ab*, *a'b'*. This was placed between

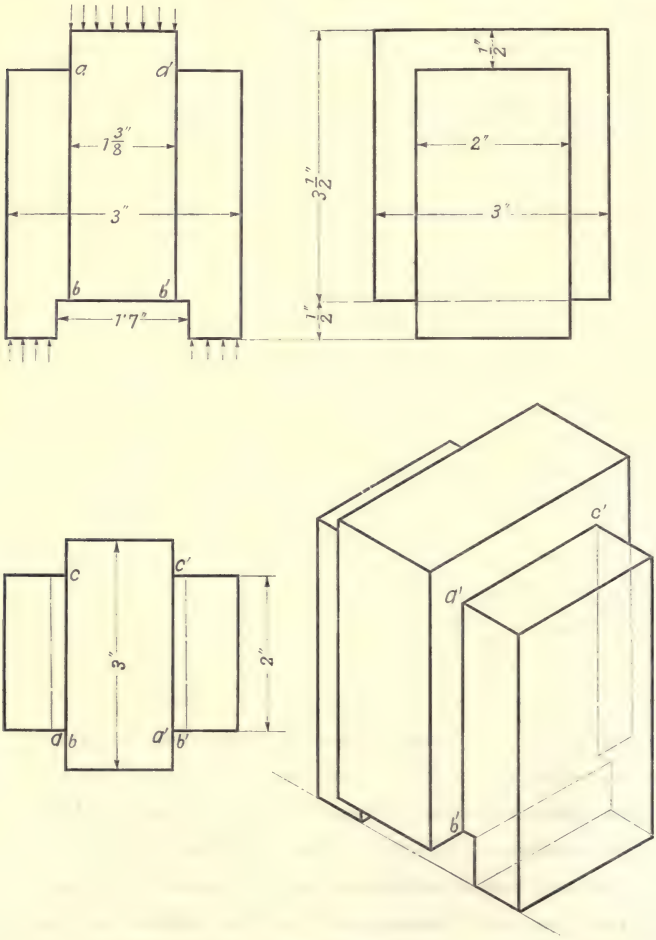


FIG. 1.

the compression plates of a hydraulic press and the pressure was gradually applied on the top and bottom of the test piece by means of a small hand pump, till the piece fractured. The total breaking load is the intensity of the pressure indicated by the attached pressure gauge multiplied by the sectional area of the plunger of the press.

We intended to shear off the test pieces over the vertical sections *abc* and *a'b'c'*. Out of 19 pieces tested, 8 gave way by shearing over one of the vertical planes as had been expected, but the remaining 11 gave way by a kind of bending action, the fracture always taking place at the bottom under tension.

Considering the middle portion of each test piece as a beam, the width is 1·76 times the span and the depth is 2·06 times the span. With such an abnormally large depth, the bending action is still great enough to cause actual fracture of the piece, owing to the very low tensile strength of timber in a direction normal to the fibre. The area of vertical section *abc* or *a'b'c'* being 6 sq. inches, the *average* ultimate shearing strength in lbs. per sq. inch is

$$f_s = \frac{1}{12}(\textit{Total breaking load in lbs.}).$$

This is given in the sixth column of the annexed tables.

Considering the test piece as a beam supported at both ends and loaded uniformly over a portion of the span, the maximum bending moment occurring at the centre of the beam is

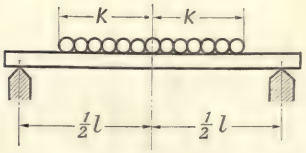


Fig. 2.

$$M_{max.} = \frac{1}{4} W l \left(1 - \frac{k}{l}\right),$$

where W is the total load on the beam in lbs. Inserting the dimensions of the beam given in Fig. 1, we have

$$M_{max.} = \frac{1}{4} \times W \times 1{\cdot}7\left(1 - \frac{1{\cdot}375}{2 \times 1{\cdot}7}\right)$$

$$= 0{\cdot}253\ W.$$

But the usual formula for the strength of a beam is

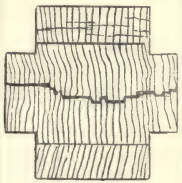
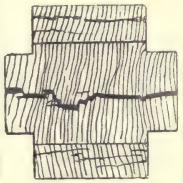
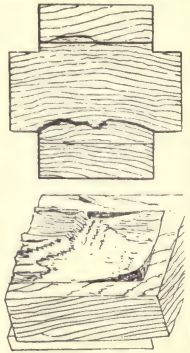
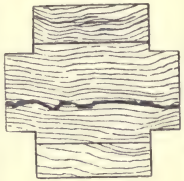
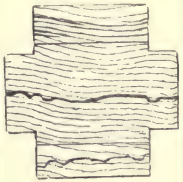
$$M = f_b \frac{I}{y},$$

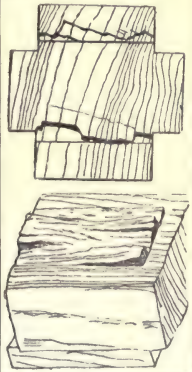
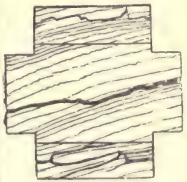
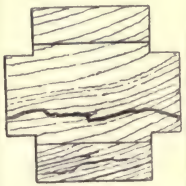
from which the coefficient of bending strength is obtained, thus

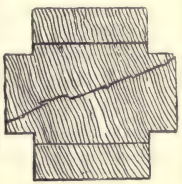
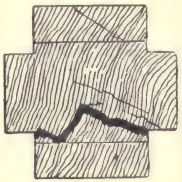
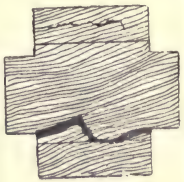
$$f_b = \frac{yM}{I} = \frac{{\cdot}253W}{\frac{1}{6}\,b\,h^2} = \frac{6 \times {\cdot}253W}{3 \times (3{\cdot}5)^2}$$

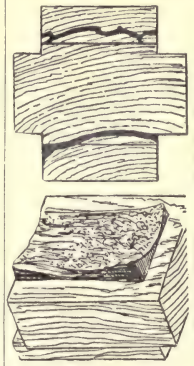
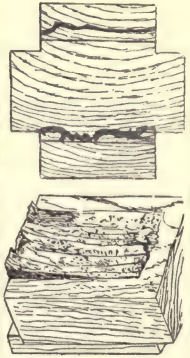
$$= {\cdot}0413W.$$

This is given in the fifth column of the annexed tables.

〃	〃	〃	〃	Sugi. 杉	Name.
4	3	2	5	1	No. marked on test piece.
across the grain.	across the grain.	along the grain.	along the grain.	along the grain.	Surface of fracture.
9,860	10,300	9,410	9,860	10,300	Total breaking load in lbs.
410	430	not less than 390	410	430	Ultimate bending strength f_b in lbs./□″.
not less than 821	not less than 859	784	not less than 821	not less than 859	Ultimate shearing strength f_s in lbs./□″.
					Sketch of bottom face and of fractured surface.

〃	〃	〃	〃	Matsu. 松	Name.
5	3	2	4	1	No. marked on test piece.
across the grain.	across the grain.	across the grain.	along the grain.	along the grain.	Surface of fracture.
13,000	9,860	11,650	11,650	12,540	Total breaking load in lbs.
540	not less than 410	480	480	520	Ultimate bending strength f_b in lbs./□″.
not less than 1,083	821	not less than 971	not less than 971	not less than 1,050	Ultimate shearing strength f_s in lbs./□″.
					Sketch of bottom face and of fractured surface.

〃	〃	〃	Hinoki. 檜	Name.
4	3	2	1	No. marked on test piece.
across the grain.	across the grain.	across the grain.	along the grain.	Surface of fracture.
11,200	11,200	11,200	11,650	Total breaking load in lbs.
460	460	not less than 460	not less than 480	Ultimate bending strength f_b in lbs./□″.
not less than 933	not less than 933	933	970	Ultimate shearing strength f_s in lbs./□″.
				Sketch of bottom face and of fractured surface.

〃	〃	〃	〃	Keyaki. 欅	Name.
4	5	3	2	1	No. marked on test piece.
across the grain.	across the grain.	across the grain.	along the grain.	along the grain.	Surface of fracture.
18,370	18,370	18,820	17,920	20,160	Total breaking load in lbs.
760	not less than 760	not less than 780	not less than 740	not less than 830	Ultimate bending strength f_b in lbs./□″.
not less than 1,531	1,531	1,568	1,493	1,680	Ultimate shearing strength f_s in lbs./□″.
					Sketch of bottom face and of fractured surface.

Strictly speaking each test piece is in the condition of a beam supported at the ends by distributed reactions and loaded uniformly over a portion of the span. Supposing the reactions to be uniform over the portions supported, the bending moment at any transverse section may be found as follows :—

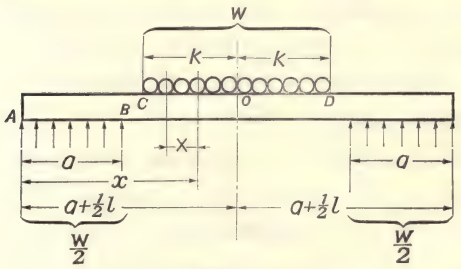


FIG. 3.

Between A and B the bending moment is

$$M_x = \int_0^x \frac{W}{2a} X dX$$

$$= \frac{1}{4a} W x^2 \quad \ldots\ldots (1)$$

Between B and C the bending moment is

$$M_x = \int_{x-a}^{x} \frac{W}{2a} X dX$$

$$= \frac{1}{4} W(2x-a). \quad \ldots\ldots (2)$$

Between C and D the bending moment is

$$M_x = \int_{x-a}^{x} \frac{W}{2a} X dX - \int_0^{x-(a+\frac{1}{2}l-k)} \frac{W}{2k} X dX$$

$$= \frac{1}{4} W(2x-a) - \frac{1}{4k} W\left\{x-(a+\frac{1}{2}l-k)\right\}^2$$

$$= \frac{1}{4} W\left\{(a+l-k) - \frac{1}{k}(a+\frac{1}{2}l-x)^2\right\}. \quad \ldots\ldots (3)$$

The greatest bending moment occurs at the centre of the span and its value is

$$M_{a+\frac{1}{2}l} = \frac{1}{4} W(a+l-k). \quad \ldots\ldots (4)$$

Inserting in this expression the values of the dimensions of the beam given in Fig. 1, we obtain the greatest bending moment at the centre of the span equal to

$$M_{a+\frac{1}{2}l} = \cdot 4156\, W,$$

which is 1·64 times the maximum bending moment calculated on the supposition that the reactions are concentrated at the ends of the span. The coefficients of bending strength, f_b, given in the fifth column of the tables may, therefore, be increased to 1·64 times the values there given.

As there exists bending action in the test piece, the greatest shearing stress occurring at the neutral surface may reach $1\frac{1}{2}$ times the average intensity of that stress distributed over each of vertical sections *bac* and *b'a'c'*. Though this may be true for a beam of uniform section and for a bending stress within the elastic limit, in the actual breaking test the f_s must necessarily be more nearly uniform and we think that the values of f_s given in the sixth column of the tables are safer numbers to employ in practice than those that would have been obtained from tests in which there is absolutely no bending action.

From a glance of the sketches of the bottom faces of the test pieces shown on the right hand column of the tables, it will be seen that the fracture occurs always either along the grain or at right angles to the grain, but not in any other direction.

All over the surface fractured by shearing we could observe fine powdery dust of abraded wood-particles. This was especially noticeable in the case of *keyaki*.

Further and more accurate tests are being made with test pieces in which the ratio of depth to the span is still greater than in this set of tests.

Tokyo, September, 1901.

XIX.

Strain Energy of a Beam beyond the Elastic Limit.

By

A. Inokuty, *Kogakuhakushi, Kogakushi*

and

Fuji Tanaka, *Kogakushi.*

[*The Journal of the College of Engineering, Tokyo Imperial University, Japan, Vol. II, No. 3, January*, 1905.]

In an ordinary commercial tension test of wrought iron, steel and other ductile metals, the object being to judge of the quality of the material for use in a particular construction, it is usual to determine two constants, namely, the breaking tensile strength obtained by dividing the greatest load by the original sectional area of the test bar and the total elongation of the bar of a given length expressed in percentage. If, in addition to these two numbers, another number representing the amount of work required to strain the test bar to the greatest total load, or to tear it asunder, is obtained from a diagram of load and elongation, it can be said that we are in possession of important practical knowledge of the quality of the material. In the case of

wrought iron and mild steel the amount of work required to tear a test bar asunder, greatly surpasses the elastic resilience, being sometimes five hundred times as large. Now two kinds of different materials or two specimens of the same material may have the same values of the strength constant and of elasticity constant so that their resiliences are equal; but one of them may require a larger amount of energy per cubic unit in order to destroy it than the other. As a material of construction the first is evidently more reliable than the second.

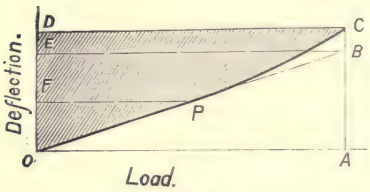


Fig. 1.

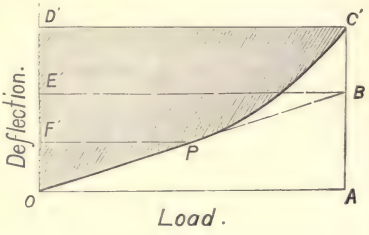


Fig. 2.

Let Figs. 1 and 2 represent load-deflection diagrams of simple rectangular beams of timber supported at the ends and loaded at the centre. In the two figures, up to the elastic limit P, the two beams are supposed to behave in exactly the same manner and also to have the same breaking load OA; but from P to C or C', which is the point of breakage, the two curves are different. The shaded area $OPC'D'$ in Fig. 2 representing the total amount of work required to break the beam is much greater than the corresponding area $OPCD$ in Fig. 1.

For a beam supported at the ends and loaded at the centre, the amount of resilience is

$$R = \frac{1}{18}\frac{f^2}{E}(\textit{volume of the beam}), \quad \ldots\ldots\ldots\ldots \quad (1)$$

in which f=the maximum bending stress at the centre of the beam,

E=the modulus of direct elasticity of the material of the beam.

Expressions for the resilience of a beam supported and loaded in any other way, or of a torsion bar or in fact of any other piece of material strained within the elastic limit are similar to the above with different numerical coefficients. Now in bending test of timber it is comparatively easy to obtain a load-deflection diagram in which the angle AOP is not very small, and to take good measurements even by means of ordinary measuring instruments of no great precision. Thus referring to Fig. 1 or 2, the value of f corresponding to the elastic limit can be scaled off with a fair accuracy from the diagram and the measured deflection OF or OF' corresponding to f enables us to calculate the value of E. The area of the triangle OPF or OPF' represents the elastic resilience R above given. If we put in (1) f equal to the so-called bending strength calculated by

$$f=\frac{3Wl}{2bh^2},$$

the result will be the resilience of the beam under the assumption that it is perfectly elastic up to the breaking point. In the figures this is represented by the area OBE or OBE'. Let this be "*hypothetical breaking resilience*" of the beam. The "*actual breaking resilience*" taking the propriety of the term for granted, is the area of the shaded portion $OPCD$ or $OPC'D'$, and since the area $OPC'D'$ is greater than the area $OPCD$, we can say that the beam of which Fig. 2 is the diagram is more reliable than the beam of which Fig. 1 is the diagram as mentioned

before. In order to express the numerical value of this reliability, we take the ratio:—

$$\frac{\textit{Actual breaking resilience}}{\textit{Hypothetical breaking resilience}}$$

and call it the "*coefficient of reliability.*" Some specimens of timbers and a few pieces of cast iron were tested by bending and the results are given in the accompanying tables.

In the fourth column the value (1) of elastic resilience in inch lbs. per cub. inch of the beam at elastic limit is given. The point of elastic limit can not be definitely located on account of the vagueness of the point at which the stress-strain curve begins to deviate from a strainght line.

In the fifth column the value (2) of hypothetical breaking resilience in inch lbs. per cub. inch of the beam is given, and it is very easy to determine this value.

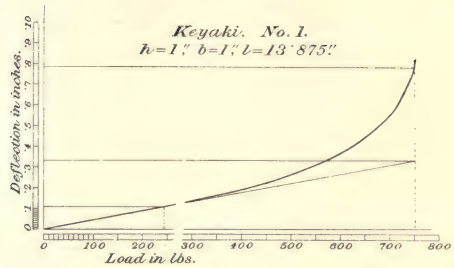
The value (3) of actual breaking resilience in inch lbs. per cub. inch is measured and calculated from the stress-strain diagrams and given in the sixth column of the tables. Strictly speaking, speed of loading is not without influence on the elongation or deformation of the test pieces, especially for timber test pieces. Hence the stress-strain diagrams of two test pieces of the same form and quality may differ to some extent according to the speed of applying the load upon it, and the area, by the same reason. But in the present experiments, practical value of the resilience is aimed at rather than refined accuracy, and the diagrams taken at ordinary speed will suffice here.

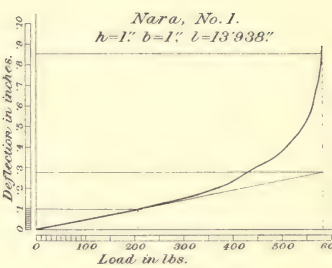
The ratio $\frac{(1)}{(2)}$ is shown in seventh column of the tables. This ratio represents the degree of reliability in some sense; for, machines or structures are strained always within elastic limit in

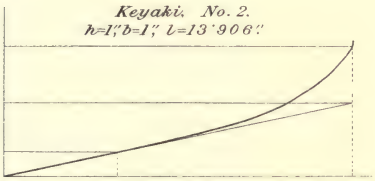
Bending Test of Timbers.

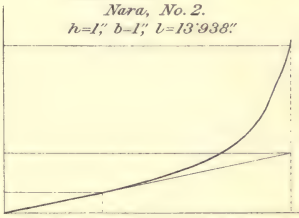
Stress-Strain Diagrams.

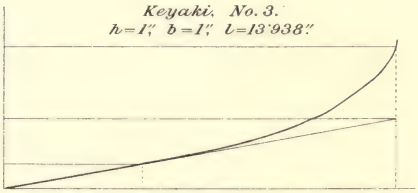
I.

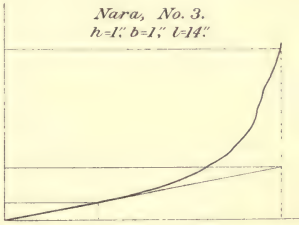


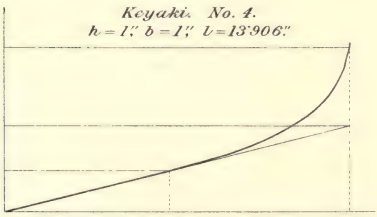


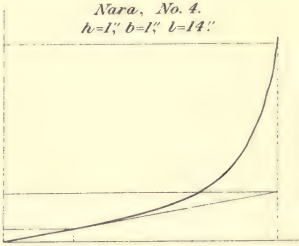


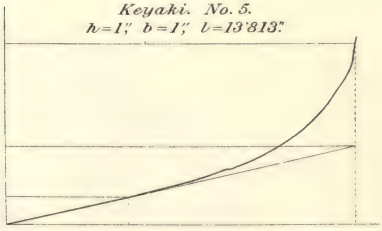


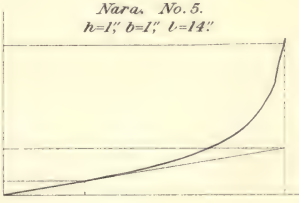








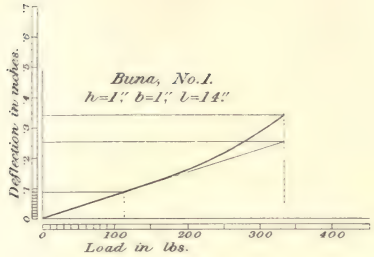


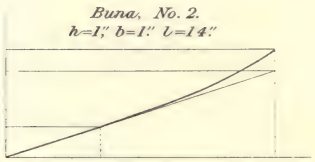


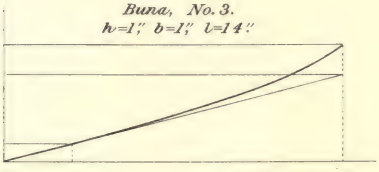
Bending Test of Timbers.

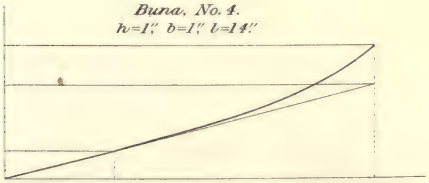
Stress-Strain Diagrams.

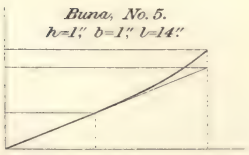
II.

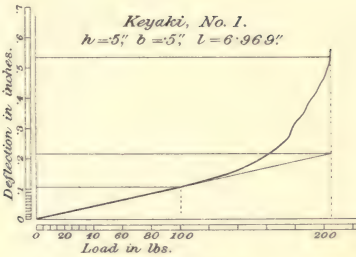


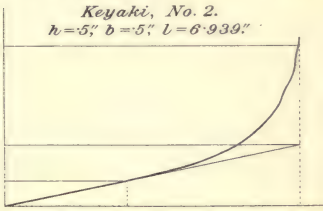


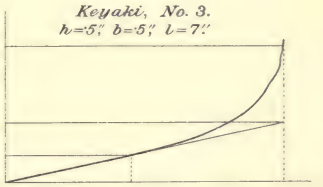


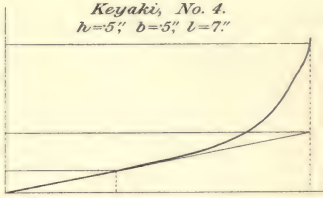


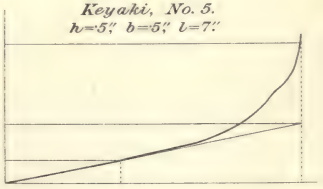








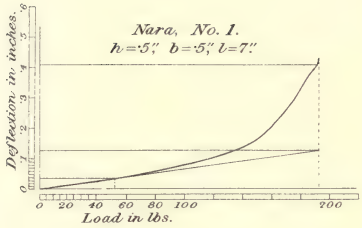


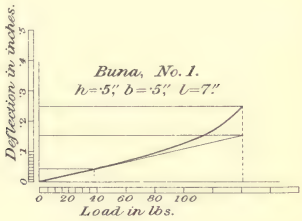


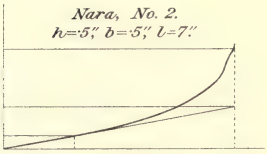
Bending Test of Timbers.

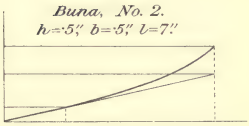
Strees-Strain Diagrams.

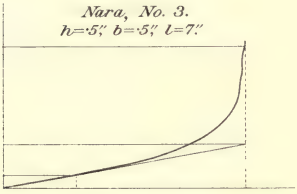
III.

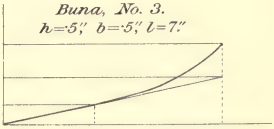


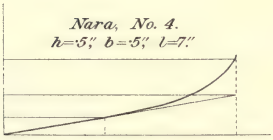


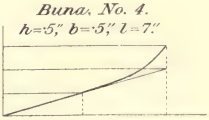


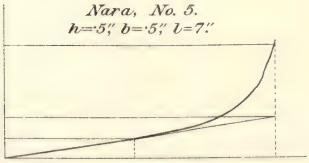


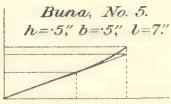








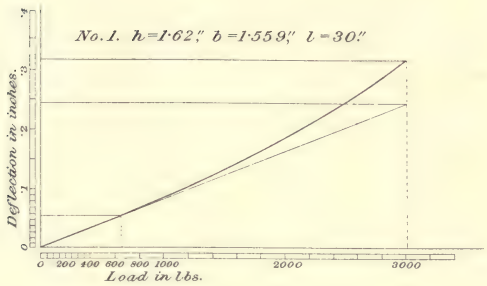


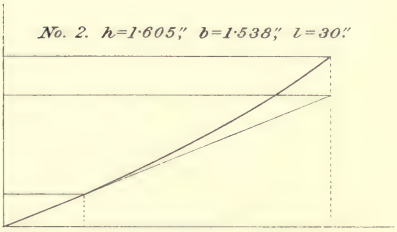


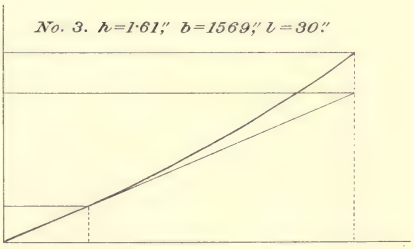
Bending Test of Cast-iron.

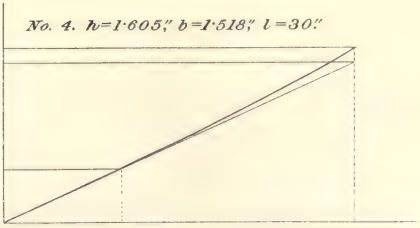
Strees-Strain Diagrams.

IV.









an ordinary working condition, and consequently the higher the elastic limit for a given breaking strength, the more reliable is the material with reference to its resistance to permanent injury, the ratio $\frac{(1)}{(2)}$ being greater in this case than when otherwise.

The ratio $\frac{(3)}{(2)}$ is given in the last column of the tables, and this is the coefficient of reliability as already described. The greater this ratio, the more resistant against blows and shocks.

From the tables we may say that *nara* is more reliable than *keyaki*, and *buna* is very poor as in the case of cast iron and can not stand much against shocks and vibrations.

It may be remarked that the three values (1), (2), and (3) of resilience in inch lbs. per cub. inch of $\cdot 5'' \times \cdot 5'' \times 7''$ sizes are greater than those of greater sizes, namely, $1'' \times 1'' \times 14''$. Such will always be the case for timber; because as is fairly well known the coefficient of bending strength f_b is greater for smaller sizes of test pieces than for larger ones.

April, 1902.

Bending Test of Timbers.

Supported at both ends, and loaded at the centre.

Name of timber.	No. marked on test piece.	Size. $h \times b \times l$ in inches.	Resilience in inch-lbs. per cub. in. $= \frac{R}{vol.}$ at elastic limit. (1)	hypothetical breaking resilience. (2)	actual breaking resilience. (3)	$\frac{(1)}{(2)}$	Coefficient of reliability. $\frac{(3)}{(2)}$
Keyaki	1	1×1×13·875	·965	8·96	30·5	·108	3·40
	2	1×1×13·906	·979	8·79	20·6	·111	2·35
	3	1×1×13·938	1·110	9·33	26·3	·119	2·82
	4	1×1×13·906	2·41	10·4	28·0	·231	2·68
	5	1×1×13·813	1·14	9·46	32·0	·120	3·38
		Mean	1·3	9·4	27·	·14	2·9
Nara	1	1×1×13·938	·723	5·94	27·4	·122	4·62
	2	1×1×13·938	·681	6·05	25·6	·113	4·24
	3	1×1×14	·587	5·00	25·2	·117	5·03
	4	1×1×14	·328	4·81	30·7	·068	6·38
	5	1×1×14	·419	4·84	24·1	·087	4·97
		Mean	·55	5·3	27·	·10	5·0
Buna	1	1×1×14	·356	2·97	4·74	·120	1·59
	2	1×1×14	·438	3·63	5·31	·121	1·46
	3	1×1×14	·203	4·86	7·54	·042	1·55
	4	1×1×14	·488	5·61	9·55	·087	1·70
	5	1×1×14	·543	2·73	3·63	·199	1·33
		Mean	·41	4·0	6·2	·115	1·5

Bending Test of Timbers (Continued.)

Name of timber.	No. marked on test piece.	Size. $h \times b \times l$ in inches.	Resilience in inch lbs. per cub. in. $= \frac{R}{vol.}$ at elastic limit. (1)	hypothetical breaking resilience. (2)	actual breaking resilience. (3)	$\frac{(1)}{(2)}$	Coefficient of reliability. $\frac{(3)}{(2)}$
Keyaki	1	·5 × ·5 × 6·969	2·97	12·8	47·7	·231	3·71
	2	·5 × ·5 × 6·938	2·01	11·8	47·4	·170	4·01
	3	·5 × ·5 × 7	2·18	10·8	35·9	·202	3·33
	4	·5 × ·5 × 7	1·56	12·0	43·2	·130	3·59
	5	·5 × ·5 × 7	1·69	11·1	39·6	·153	3·57
		Mean	2·1	11·5	43·	·175	3·6
Nara	1	·5 × ·5 × 7	·446	6·71	34·3	·066	5·10
	2	·5 × ·5 × 7	·599	5·93	21·4	·101	3·61
	3	·5 × ·5 × 7	·616	6·61	35·3	·093	5·35
	4	·5 × ·5 × 7	1·10	5·93	15·3	·185	2·58
	5	·5 × ·5 × 7	1·54	6·80	29·9	·226	4·40
		Mean	·86	6·4	27·	·135	4·2
Buna	1	·5 × ·5 × 7	·731	6·14	12·7	·119	2·06
	2	·5 × ·5 × 7	·546	6·42	12·7	·085	1·99
	3	·5 × ·5 × 7	1·17	6·63	14·7	·177	2·21
	4	·5 × ·5 × 7	1·115	4·82	8·86	·291	1·84
	5	·5 × ·5 × 7	1·23	3·54	4·53	·347	1·28
		Mean	·96	5·5	10·5	·20	1·90

Bending Test of Cast Iron.

Supported at both ends, and loaded at the centre.

Cast iron	1	1·62 × 1·559 × 30	·231	4·84	6·96	·048	1·44
	2	1·605 × 1·538 × 30	·224	3·84	5·52	·058	1·43
	3	1·61 × 1·569 × 30	·282	4·72	6·62	·060	1·40
	*4	1·605 × 1·518 × 30	·608	5·21	6·01	·117	1·15
		Mean	·34	4·7	6·3	·071	1·35
		(Mean of upper three)	·25	4·5	6·4	·055	1·40

* Strained twice beyond the elastic limit, and then broken.

XX.

Universal Repetitive Bending Test.

By

A. Inokuty, *Kogakuhakushi, Kogakushi*

and

Fuji Tanaka, *Kogakushi.*

[*The Journal of the College of Engineering, Tokyo Imperial University, Japan, Vol. II, No.* 3, *January*, 1905.]

XXI.

Results of Tests of Pressure Gauges.

By

A. Inokuty, *Kogakuhakushi, Kogakushi*

and

Fuji Tanaka, *Kogakushi.*

[*The Jouranl of the College of Engineering, Tokyo Imperial University, Japan, Vol. II, No.* 3, *January*, 1905.]

XXII.

Theory of Ordinary Centrifugal Pumps and of a New Centrifugal Pump having Divergent Vortex Chamber provided with Guide Vanes for producing Forced Vortex.

[*The Journal of the College of Engineering, Tokyo Imperial University, Japan, Vol. II, No.* 4, *March* 1905. *The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. VIII, No.* 12, *May* 1905.]

A few hydraulic principles relating to centrifugal pumps will be considered first.

Radiating Current. In Fig. 1 let $AABB$ represent a space bounded by two horizontal walls and lying between two cylindrical surfaces with a common vertical axis ZOZ and of radii r_2 and r_3. A current of water moving toward or away from the axis ZOZ in radial directions in the space $AABB$ constitutes a radiating current. It is supposed that there is no external force acting on the current except the gravity and a uniform fluid pressure on each of the cylindrical surfaces. Let u_2, u, u_3 be the radial velocities of particles of water at radii r_2, r, r_3 respectively. From the principle of continuity, we have

$$2\pi rbu = 2\pi r_2 b_2 u_2$$
$$= 2\pi r_3 b_3 u_3$$
$$= Q,$$

in which Q is the quantity of the flow across any co-axial cylindrical surface of radius r, in units of volume per unit time. The motion of water being supposed steady, this flow Q is constant with reference to the radius r as well as with reference to time.

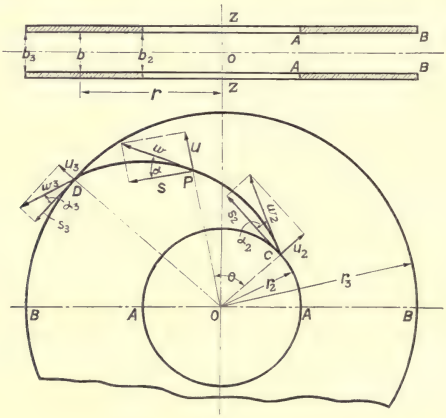


Fig. 1.

Hence $ru = r_2 u_2 = r_3 u_3 = const.,$ ..... (1)

that is, the radial velocity is inversely as the distance from the axis.

Free Circular Vortex. Referring to Fig. 1 let the particles of water in the space $AABB$ revolve in circular current round the vertical axis ZOZ. There being supposed to exist a very slow radial movement, a particle of water in passing from one circular current to the next circular current will assume the

velocity proper to the current entered by the particle. As in the case of radiating current considered above, the only forces acting on the water in the space $AABB$ are the gravity and a uniform fluid pressure on each of the cylindrical surfaces of radii r_2 and r_3. It is this condition which constitutes a free circular vortex. Putting s_2, s, s_3, for the tangential velocities at radii r_2, r, r_3 respectively and ρ for the weight of water per cubic unit, we have by Bernoulli's theorem

$$\frac{p}{\rho}+\frac{s^2}{2g}=\frac{p_2}{\rho}+\frac{s_2^2}{2g}=\frac{p_3}{\rho}+\frac{w_3^2}{2g}=const. \quad \ldots\ldots\ldots\ldots \quad (2)$$

The centrifugal force of a particle of water at radius r, of radial thickness dr, of axial depth b, and of circumferential width $rd\theta$ is

$$\frac{\rho}{g} r\, d\theta\, b\, dr \frac{s^2}{r}.$$

This is equal and opposite to the difference of pressures on the cylindrical faces, namely,

$$r\, d\theta\, b\, dp.$$

Hence
$$\frac{dp}{dr}=\frac{\rho}{g}\frac{s^2}{r}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

Differentiating equation (2), we have

$$\frac{1}{\rho}\frac{dp}{dr}+\frac{s}{g}\frac{ds}{dr}=0.$$

Eliminating $\frac{dp}{dr}$ between this and (3), and removing irrelevant factors, we have

$$\frac{s}{r}+\frac{ds}{dr}=0.$$

Hence
$$\frac{dr}{r}+\frac{ds}{s}=0.$$

Integrating we obtain

$$rs = const. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Thus in a free circular vortex the velocity of revolution of a particle of water is inversely as the radius.

Free Spiral Vortex. Referring to Fig. 1 let there be a free spiral vortex in the space *AABB*, that is, a current along a spiral path *CPD*, the acting forces being the gravity and a uniform fluid pressure on the cylindrical surfaces of radii r_2 and r_3. Such a current is evidently the resultant of a radiating current and a free circular current. Hence if w be the velocity of a particle of water at radius r in a free spiral vortex, we have

$$w = \sqrt{u^2 + s^2},$$

which by (1) and (4) becomes

$$w = \frac{const.}{r}$$

or

$$wr = const. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

Hence the velocity of a particle of water in a free spiral vortex is inversely as the radius. The inclination of the path of a particle to the radius vector from O is $\frac{\pi}{2} - \alpha$ and its value is given by

$$\tan \alpha = \frac{u}{s} = const. \quad \ldots\ldots\ldots\ldots \quad (6)$$

The path of the particle, therefore, makes a constant angle with the radius vector and is an equiangular or logarithmic spiral. The equation of the path can be found as follows:—

$$u = \frac{dr}{dt} \quad \text{and} \quad s = r\frac{d\theta}{dt},$$

therefore $$\cot \alpha = \frac{s}{u} = r\frac{d\theta}{dt} \div \frac{dr}{dt} = r\frac{d\theta}{dr},$$

$$d\theta = \frac{dr}{r}\cot \alpha.$$

Integrating, we have

$$\theta = \cot \alpha \log r + const.$$

When $\theta = 0, \quad r = r_2$

and therefore $0 = \cot \alpha \log r_2 + const.$

Hence $$r = r_2 e^{\theta \tan \alpha} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

The gain of pressure head in passing from the inner cylindrical surface to the outer of a free spiral vortex between two horizontal planes is found as follows. By Bernoulli's theorem,

$$\frac{p_2}{\rho} + \frac{w_2^2}{2g} = \frac{p_3}{\rho} + \frac{w_3^2}{2g}.$$

Hence $$\frac{1}{\rho}(p_3 - p_2) = \frac{1}{2g}(w_2^2 - w_3^2).$$

But $$w_2 r_2 = w_3 r_3.$$

Hence $$\frac{1}{\rho}(p_3 - p_2) = \frac{w_2^2}{2g}\left\{1 - \left(\frac{r_2}{r_3}\right)^2\right\}. \quad \ldots\ldots\ldots \quad (8)$$

We shall now consider another kind of vortex. In Fig. 2, let *ZOZ* represent a vertical axis and let *AABB* be a space formed by the revolution of two plane curves *AB*, *AB* about the vertical axis *ZOZ*, the two curves being symmetrical about a horizontal axis *XOX*, so that the space *AABB* diverges outwards symmetrically about the vertical axis *ZOZ*. The space *AABB* lies between two cylindrical surfaces of radii r_2 and r_3, with *ZOZ* for their common axis.

Divergent Radiating Current. A radiating current in the space *AABB*, Fig. 2, gives, by the principle of continuity, the following equations,

$$rbu = r_2 b_2 u_2 = r_3 b_3 u_3$$

$$= \frac{Q}{2\pi} = const. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

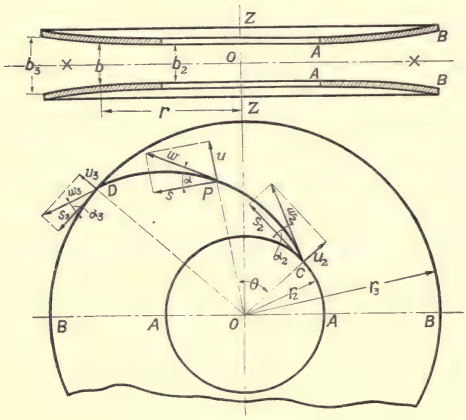


FIG. 2.

Divergent Free Circular Vortex. Referring to Fig. 2 let particles of water in the space *AABB* revolve in circular current round the axis *ZOZ*, with a very slow radial movement so that a particle of water in passing from one circular current to the next circular current will assume the velocity proper to the current entered by the particle. This will happen if the only acting forces be the gravity and a uniform fluid pressure on each of the cylindrical surfaces of radii r_2 and r_3. It is this condition which constitutes a free circular vortex. Consider a thin circular current of radius r, of radial thickness dr, and of axial depth b,

the velocity of revolution of a particle of water in the current being s. The centrifugal force of an element, having circumferential width $rd\theta$, of this current is

$$\frac{\rho}{g} r\, d\theta\, b\, dr \frac{s^2}{r}.$$

This is equal and opposite to the difference of pressures on the two cylindrical faces of the element, which is

$$r\, d\theta\, b\, dp.$$

Hence
$$\frac{dp}{dr} = \frac{\rho}{g}\frac{s^2}{r}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

Now by Bernoulli's theorem

$$\frac{p}{\rho} + \frac{s^2}{2g} = \frac{p_2}{\rho} + \frac{s_2^2}{2g} = const.,$$

which by differentiation becomes

$$\frac{1}{\rho}\frac{dp}{dr} + \frac{s}{g}\frac{ds}{dr} = 0.$$

Eliminating $\frac{dp}{dr}$ between this and (10), and removing irrelevant factors, we obtain

$$\frac{dr}{r} + \frac{ds}{s} = 0.$$

Integrating we obtain

$$rs = const. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

Divergent Free Spiral Vortex. Referring to Fig. 2, let there be a free spiral vortex in the space $AABB$. The motion is evidently the combination of a radiating current and a free circular vortex as above given; we have therefore the velocity of the current equal to

$$w=\sqrt{u^2+s^2},$$

which by (9) and (11) becomes

$$w=\frac{r_2}{r}\sqrt{\left(\frac{b_2}{b}\right)^2 u_2^2+s_2^2}, \quad \ldots\ldots\ldots \quad (12)$$

therefore

$$w_3=\frac{r_2}{r_3}\sqrt{\left(\frac{b_2}{b_3}\right)^2 u_2^2+s_2^2}, \quad \ldots\ldots\ldots \quad (13)$$

and

$$\tan\alpha=\frac{u}{s}=\frac{b_2}{b}\frac{u_2}{s_2}$$

$$=\frac{b_2}{b}\tan\alpha_2 \quad \ldots\ldots\ldots \quad (14)$$

therefore

$$\tan\alpha_3=\frac{b_2}{b_3}\tan\alpha_2. \quad \ldots\ldots\ldots \quad (15)$$

Forced Vortex. A forced vortex is one in which the velocity of revolution of the particles follows any law different from that of a free vortex. Referring again to Fig. 2 we consider a forced or constrained spiral vortex in which the velocity of revolution follows a law expressed by

$$rbs=const. \quad \ldots\ldots\ldots \quad (16)$$

Combining this with the equation (9), we have

$$\tan\alpha=\frac{u}{s}=const. \quad \ldots\ldots\ldots \quad (17)$$

Hence in the divergent space, the forced spiral vortex above assumed is obtained by constraining the particles of water to move along equiangular spirals by means of a number of guide vanes. Since the resultant velocity is $w=\sqrt{u^2+s^2}$, we have by (9) and (16)

$$rbw=const. \quad \ldots\ldots\ldots \quad (18)$$

Referring once more to Fig. 2, in any spiral vortex whether free or forced,

$$rbu = r_2 b_2 u_2 = r_3 b_3 u_3$$

$$= \frac{Q}{2\pi} = const.,$$

and since

$$u = w \sin \alpha, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (19)$$

we have

$$brw \sin \alpha = b_2 r_2 w_2 \sin \alpha_2$$

$$= b_3 r_3 w_3 \sin \alpha_3$$

$$= const. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (20)$$

Hence by giving suitable values to b and α, it is possible to obtain any desired radial velocity u and resultant velocity w.

Gain of Pressure Head in Vortex. In any spiral vortex whether free or forced existing within the space $AABB$, Fig. 2, if there be no external force exerted within the space $AABB$, except that of gravity, the theorem of Bernoulli holds good; thus

$$\frac{p_2}{\rho} + \frac{w_2^2}{2g} = \frac{p_3}{\rho} + \frac{w_3^2}{2g}.$$

The gain of pressure head in passing from the inner cylindrical surface of the vortex to the outer is therefore

$$\frac{1}{\rho}\left(p_3 - p_2\right) = \frac{1}{2g}\left(w_2^2 - w_3^2\right),$$

which by (20) becomes

$$\frac{1}{\rho}\left(p_3 - p_2\right) = \frac{w_2^2}{2g}\left\{1 - \left(\frac{r_2 b_2 \sin \alpha_2}{r_3 b_3 \sin \alpha_3}\right)^2\right\}. \quad \ldots\ldots\ldots \quad (21)$$

Loss of Head due to Sudden Enlargement of Section. At a sudden enlargement of the sectional area of a steady stream flowing along a straight pipe there is a certain amount of loss of head owing to the creation of eddies at and beyond the enlarged section. This loss of head is equal to the head due to the change of velocity. The proof of this is, however, not very satisfactory and must be taken partly experimental. (See Prof. Merriman's Treatise on Hydraulics, Eighth Edition, Art. 74.)

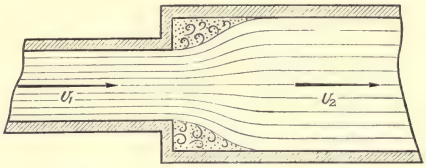


Fig. 3.

Thus supposing the pipe to be horizontal, if the velocity v_1 of the water changes to a smaller velocity v_2 as shown in longitudinal section in Fig. 3, the loss of pressure head at a section where the velocity is sensibly uniform and equal to v_2 is

$$\frac{(v_1-v_2)^2}{2g}.$$

Let p_1 and p_2 be the pressures at the sections where the velocities are v_1 and v_2 respectively, then by Bernoulli's theorem we have

$$\frac{p_1}{\rho}+\frac{v_1^2}{2g}=\frac{p_2}{\rho}+\frac{v_2^2}{2g}+\frac{(v_1-v_2)^2}{2g}.$$

Hence the gain of pressure head in passing from the region of velocity v_1 to the region of velocity v_2 is

$$\frac{1}{\rho}(p_2-p_1)=\frac{1}{2g}\left\{v_1^2-v_2^2-(v_1-v_2)^2\right\}$$

$$=\frac{1}{2g}2\,v_2(v_1-v_2).$$

Supposing the higher velocity v_1 to be constant, the gain of pressure head will be maximum when $v_2=\frac{1}{2}v_1$ and

$$max.\ value\ of\ \frac{1}{\rho}(p_2-p_1)=\frac{1}{2}\frac{v_1^2}{2g},\quad \ldots\ldots\ldots\ (22)$$

that is, one half the head due to the higher velocity. In this case the loss of pressure head is

$$\frac{(v_1-\frac{1}{2}v_1)^2}{2g},$$

i.e.,

$$\frac{1}{4}\frac{v_1^2}{2g}.$$

It is thus seen that by making the sectional area of the enlarged part equal to twice that of the smaller part, one half of the head due to the higher velocity is restored as pressure, one fourth is wasted in creating eddies and the remaining one fourth is still in the form of velocity.

Sudden Change of Velocity both in Magnitude and Direction. We now consider another case of sudden change of velocity in

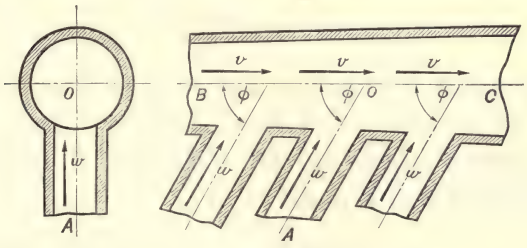


FIG. 4.

a steady flow. In Fig. 4, AO represents one of a very large number of small streams flowing with a given velocity w and making a given angle ϕ with the direction of a main stream BOC, in which the velocity is less than w and is denoted by v. The stream AO meeting with the stream BOC, there is a sudden change of velocity, first in the direction perpendicular to BOC, of magnitude $w \sin \phi$, and secondly in the direction BOC, of magnitude $w \cos \phi - v$. The loss of head due to the sudden change of velocity is therefore

$$\frac{1}{2g}\left\{(w \sin \phi)^2 + (w \cos \phi - v)^2\right\},$$

i.e.,

$$\frac{1}{2g}(w^2 + v^2 - 2wv \cos \phi).$$

Let p be the pressure in the stream AO and p' that in the stream BOC; then by Bernoulli's theorem

$$\frac{p}{\rho} + \frac{w^2}{2g} = \frac{p'}{\rho} + \frac{v^2}{2g} + \frac{(w \sin \phi)^2}{2g} + \frac{(w \cos \phi - v)^2}{2g},$$

and therefore the gain of pressure head in passing from AO to BOC is

$$\frac{1}{\rho}(p' - p) = \frac{1}{2g}\left\{w^2 - v^2 - (w \sin \phi)^2 - (w \cos \phi - v)^2\right\}$$

$$= \frac{1}{2g} 2v (w \cos \phi - v). \quad \ldots\ldots\ldots\ldots \quad (23)$$

Suppose that the velocity w and the angle ϕ are both given but that the sectional area of the channel BOC can be varied at will so that v may be regarded as variable. In this case the gain of pressure head is maximum when

$$v = \frac{1}{2} w \cos \phi \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (24)$$

and the corresponding maximun gain of pressure is

$$\frac{1}{2}\frac{(w\cos\phi)^2}{2g}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (25)$$

Now assuming that the velocity v in the canal BOC cannot be restored as pressure, the total loss of pressure head consists of three parts and is expressed by

$$h_0=\frac{1}{2g}\left\{(w\sin\phi)^2+(w\cos\phi-v)^2+v^2\right\}, \quad \ldots\ldots\ldots\ldots \quad (26)$$

which will be the least when $v=\frac{1}{2}w\cos\phi$ and the corresponding minimum value of h_0 is

$$min.\ h_0=\frac{1}{2g}\left\{(w\sin\phi)^2+\frac{1}{2}(w\cos\phi)^2\right\}. \quad \ldots\ldots\ldots \quad (26')$$

General Theory of Centrifugal Pumps. A general theory of centrifugal pumps will now be considered, neglecting for the present the effect of hydraulic friction. In order to simplify our reasoning it is assumed in all the following pages that the pressure of water is uniform round the inner and the outer circumference of the pump.

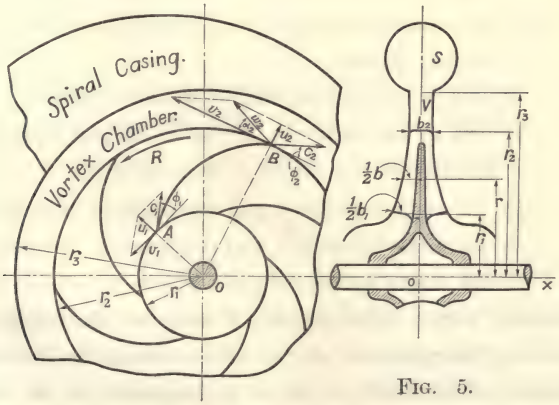


FIG. 5.

In Fig. 5 AB represents one blade of revolving disk or impeller, revolving in the direction of the arrow R round the axis XOX of the pump shaft. It is usually assumed that the direction of flow of water before entering the revolving disk is radial; this assumption is, however, slightly inaccurate, because owing to the friction of the pump shaft and of the wheel boss, the mass of water surrounding these parts will be dragged round them bodily in the direction of revolution of the pump disk. Although the exact amount of this circumfluent motion cannot be found, it is certain that it will be greater the higher the speed of rotation of the parts round the pump shaft and the lower the axial velocity of the water before entering the pump disk. In the following theory the slight inaccuracy above mentioned is neglected so that the direction of the absolute velocity of water before entering the revolving disk is regarded as radial.

Let w_1 and w_2 be the absolute velocities of water at the inlet and outlet surfaces of the revolving disk or impeller respectively,

c_1 and c_2 the corresponding velocities relatively to the moving blades,

u_1 and u_2 the corresponding radial velocities,

v_1 and v_2 the velocities of the inner and outer circumferences of the impeller respectively,

ϕ_1 the angle between c_1 and the negative direction of v_1,

ϕ_2 the angle between c_2 and the negative direction of v_2,

a_2 the angle between w_2 and v_2.

In order that a direct impact of water on the blades of the impeller may be prevented at the inlet surface, the direction of the relative inflow velocity c_1 must be tangential to the first ele-

ment of the blade at A, Fig. 5. With the assumption above made, this condition is expressed by

$$c_1^2 = u_1^2 + v_1^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (27)$$

or by $$u_1 = c_1 \sin \phi_1 \quad \text{and} \quad v_1 = c_1 \cos \phi_1. \quad \ldots\ldots\ldots \quad (27')$$

We may here add that the velocity w_1 is equal to u_1 and the angle α_1 between w_1 and v_1 is equal to 90°. The absolute velocity of water with which it leaves the impeller at B, Fig. 5, is expressed by

$$w_2^2 = c_2^2 + v_2^2 - 2c_2 v_2 \cos \phi_2. \quad \ldots\ldots\ldots\ldots\ldots \quad (28)$$

Now let h_a be the height of water barometer, h_1 the height of the suction estimated from the lower water-level to the mean level of the inlet surface of the impeller and F_1 the head due to frictional and other hydraulic resistances in the suction pipe and the passages in the pump up to the inlet surface of the impeller. The absolute pressure of water just after entering the impeller at A is less than the atmospheric pressure and is expressed by

$$\frac{p_1}{\rho} = h_a - h_1 - F_1 - \frac{w_1}{2g}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (29)$$

where ρ is the weight of water per cubic unit. Employing corresponding notations with suffix 2, the absolute pressure of water just before leaving the impeller at B is expressed by

$$\frac{p_2}{\rho} = h_a + h_2 + F_2 - \frac{w_2}{2g}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (30)$$

in which h_2 is the height of delivery estimated from the mean level of the outlet surface of the impeller to the level of delivery water and F_2 represents the head lost in the outflow velocity w_2 and also in the frictional and other hydraulic resistances in

the delivery pipe, the volute or spiral casing, and the vortex chamber, if there is one. The water in passing through the impeller from A to B has its velocity head increased by the centrifugal force by the amount

$$\frac{v_2^2}{2g}-\frac{v_1^2}{2g}.$$

The general equation for the action of the impeller on the water is

$$\frac{p_2}{\rho}+\frac{c_2^2}{2g}+F_w-\left(\frac{p_1}{\rho}+\frac{c_1^2}{2g}\right)=\frac{v_2^2}{2g}-\frac{v_1^2}{2g},^{*} \quad \ldots\ldots\ldots \quad (31)$$

in which F_w represents the head lost in friction of water in passing through the passages of the wheel. Substituting the values of p_1, p_2, c_1, and c_2 given by equations (27) to (30), we obtain

$$h_1+h_2+F_1+F_2+F_w=\frac{1}{g}v_2(v_2-c_2\cos\phi_2).$$

Let the total actual lift be $h=h_1+h_2$ and put $F=F_1+F_2+F_w$. It is easy to see that $v_2-c_2\cos\phi_2$ is the tangential component of the absolute outflow velocity w_2 and

$$w_2\cos\alpha_2=v_2-c_2\cos\phi_2 \quad \ldots\ldots\ldots\ldots\ldots \quad (32)$$

also

$$w_2\sin\alpha_2=u_2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (33)$$

The above equation for the gross lift of the pump may be written as follows:—

---

* This equation is easily obtained from the following general principle:—

If a steady motion of water takes place through a pipe of any form attached to a rotating shaft, a modified form of Bernoulli's theorem can be found. Thus:—

$$h+\frac{p}{\rho}+\frac{c^2}{2g}=constant+\frac{v^2}{2g},$$

where c is the velocity of water at a section of the pipe under consideration, relatively to the moving pipe, v the velocity of rotation of the section, and the rest of the notation the same as those hitherto employed. The quantity $v^2\div 2g$ is sometimes called the "head due to centrifugal force."

$$h+F=\frac{1}{g}v_2(v_2-c_2\cos\phi_2) \quad \ldots\ldots\ldots\ldots\ldots \quad (34)$$

$$=\frac{1}{g}v_2w_2\cos a_2. \quad \ldots\ldots\ldots\ldots\ldots \quad (34')$$

Now the efficiency of the centrifugal pump is given by

$$\varepsilon=\frac{h}{h+F}=\frac{2gh}{2v_2(v_2-c_2\cos\phi_2)}$$

$$=1-\frac{gF}{v_2(v_2-c_2\cos\phi_2)}. \quad \ldots\ldots\ldots\ldots\ldots \quad (35)$$

If the loss of head due to friction in the pipes and passages of the pump be neglected, the remaining part of F will be almost solely the loss of velocity head, $(w_2^2\div 2g)$, in part or in whole. This loss of velocity head and the corresponding efficiency for various forms of centrifugal pumps that are likely to occur in practice and also for a new form of pumps constructed by the author after a principle of forced vortex motion will now be considered, neglecting for the present the effect of hydraulic friction.

Case 1. *A centrifugal pump without a vortex or whirlpool chamber, the spiral or volute casing being not well designed.*

Ready-made centrifugal pumps placed on the market at very cheap prices may be regarded to belong to Case 1 almost without exception. The radial component of absolute velocity w_2 with which the water leaves the impeller is necessarily wasted. This is equal to $w_2\sin a_2$ and also equal to $c_2\sin\phi_2$. Let the velocity of water in the spiral casing be denoted by w_4 whose direction is very nearly tangential to the outer circumference of the impeller and which is in the same sense as the velocity of that circumference. This velocity w_4 in the spiral casing is almost wholly

wasted in agitation of water. Then by equation (26), the loss of head is

$$\frac{1}{2g}\left\{(w_2 \sin a_2)^2+(w_2 \cos a_2-w_4)^2+w_4^2\right\}.$$

But w_4 is generally very small compared with $w_2 \cos a_2$ and may be neglected. Thus the whole of the absolute outflow velocity w_2 may be regarded as wasted, so that

$$F=\frac{w_2^2}{2g}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (36)$$

The efficiency of the centrifugal pump then becomes, by (35),

$$\varepsilon=1-\frac{\frac{1}{2}w_2^2}{v_2(v_2-c_2 \cos \phi_2)},$$

which by (28) becomes

$$\varepsilon=1-\frac{c_2^2+v_2^2-2c_2v_2 \cos \phi_2}{2v_2(v_2-c_2 \cos \phi_2)}$$

$$=\frac{1-z^2}{2(1-z \cos \phi_2)},$$

where z is put for $(c_2 \div v_2)$. Differentiating this equation with respect to z, we obtain

$$\frac{d\varepsilon}{dz}=\frac{z^2 \cos \phi_2-2z+\cos \phi_2}{2(1-z \cos \phi_2)^2}.$$

Equating to zero and solving for z, there is obtained, after reduction, the relation between v_2 and c_2,

$$v_2-c_2 \cos \phi_2=v_2 \sin \phi_2. \quad \ldots\ldots\ldots\ldots\ldots \quad (37)$$

Substituting this in the expression for the efficiency, we obtain

$$\textit{Maximum } \varepsilon=\frac{1}{1+\sin \phi_2}. \quad \ldots\ldots\ldots\ldots\ldots \quad (38)$$

Eliminating F and c_2 from the equation (34) by means of (36),

(28), and (37), we obtain the velocity of the outer circumference of the impeller corresponding to the maximum efficiency,

$$v_2=\sqrt{\frac{1}{2}(1+\operatorname{cosec}\phi_2)}\sqrt{2gh}$$

$$=\sqrt{\frac{2gh}{2\varepsilon\sin\phi_2}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (39)$$

Other velocities corresponding to the maximum efficiency are

$$w_2=\sqrt{\sin\phi_2}\sqrt{2gh}. \quad \ldots\ldots\ldots\ldots \quad (40)$$

$$c_2=\sqrt{\tfrac{1}{2}(\operatorname{cosec}\phi_2-1)}\sqrt{2gh}. \quad \ldots\ldots\ldots \quad (41)$$

$$u_2=w_2\sin\alpha_2=c_2\sin\phi_2$$

$$=\sqrt{\tfrac{1}{2}\sin\phi_2(1-\sin\phi_2)}\sqrt{2gh}, \quad \ldots\ldots\ldots \quad (42)$$

which is the radial component of the outflow velocity.

$$w_2\cos\alpha_2=v_2-c_2\cos\phi_2=v_2\sin\phi_2$$

$$=\sqrt{\tfrac{1}{2}\sin\phi_2(1+\sin\phi_2)}\sqrt{2gh}, \quad \ldots\ldots\ldots \quad (43)$$

which is the tangential component of the absolute outflow velocity. The direction of the absolute outflow velocity is given by

$$\tan\alpha_2=\frac{w_2\sin\alpha_2}{w_2\cos\alpha_2}=\frac{u_2}{w_2\cos\alpha_2}$$

$$=\sqrt{\frac{1-\sin\phi_2}{1+\sin\phi_2}}=\tan\frac{1}{2}\left(\frac{\pi}{2}-\phi_2\right),$$

therefore

$$\alpha_2=\frac{1}{2}\left(\frac{\pi}{2}-\phi_2\right). \quad \ldots\ldots\ldots\ldots\ldots \quad (44)$$

The accompanying Table I gives the maximum efficiency, the corresponding best velocity of the outer circumference of the impeller and two other velocities for several values of the discharge angle ϕ_2 from 10° to 90°. The increase of efficiency gained by diminishing the discharge angle ϕ_2 is quite consider-

able. It may be noticed, however, that for maximum efficiency the radial velocity of water ought to be very high for moderately small angle of discharge; in fact u_2 becomes greatest and equal to $0{\cdot}354\sqrt{2gh}$ for $\phi_2=30°$. It is, therefore, not convenient to employ the speed of maximum efficiency except for very low lifts.

Table I. *Showing Maximum Efficiency and the Corresponding Best Speeds for Case* 1.

Discharge angle ϕ_2	Maximum efficiency ε	Velocity of impeller, $v_2=\sqrt{2gh}\times$	Radial velocity, $u_2=\sqrt{2gh}\times$	Tangential velocity, $v_2\cos\alpha_2=\sqrt{2gh}\times$
90°	(·500)	1·000	·000	1·000
65°	·525	1·026	·206	·986
45°	·586	1·098	·322	·776
30°	·667	1·225	·354	·612
20°	·745	1·401	·335	·479
15°	·794	1·560	·319	·404
10°	·852	1·839	·268	·319

Case 2. *A centrifugal pump without a vortex chamber, the spiral casing being well-designed.*

In Case 2, the spiral casing is so designed that the loss of pressure head due to the sudden change of velocity of water in passing from the impeller to the spiral casing shall be a minimum. Referring to (26′), it is seen that this condition is satisfied when the velocity of water in the spiral casing is made equal to one half of the tangential component of the absolute outflow velocity, that is, when

$$w_4 = \frac{1}{2} w_2 \cos a_2, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (45)$$

and then the loss of head is

$$F = \frac{1}{2g}\left\{(w_2 \sin a_2)^2 + \frac{1}{2}(w_2 \cos a_2)^2\right\}$$

$$= \frac{1}{2g}\left\{(c_2 \sin \phi_2)^2 + \frac{1}{2}(v_2 - c_2 \cos \phi_2)^2\right\}. \quad \ldots \quad (46)$$

By (35), the efficiency then becomes

$$\varepsilon = 1 - \frac{(c_2 \sin \phi_2)^2 + \frac{1}{2}(v_2 - c_2 \cos \phi_2)^2}{2v_2(v_2 - c_2 \cos \phi_2)}$$

$$= 1 - \frac{(z \sin \phi_2)^2 + \frac{1}{2}(1 - z \cos \phi_2)^2}{2(1 - z \cos \phi_2)}. \quad \ldots\ldots \quad (47)$$

Differentiating this with respect to z and equating the result to zero, we obtain the condition for maximum efficiency,

$$1 - z \cos \phi_2 = \sin \phi_2', \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (48)$$

in which

$$\sin \phi_2' = \frac{\sin \phi_2}{(\sin^2 \phi_2 + \frac{1}{2} \cos^2 \phi_2)^{\frac{1}{2}}}, \quad \ldots\ldots\ldots\ldots \quad (49)$$

and the value of the maximum efficiency is

$$\textit{Maximum } \varepsilon = \frac{1 + \frac{1}{2} \sin \phi_2'}{1 + \sin \phi_2'}. \quad \ldots\ldots\ldots\ldots \quad (50)$$

Eliminating F and c_2 from the equation (34) by means of (46), (28), (48), and (49), we obtain the velocity of the outer circumference of the impeller corresponding to the maximum efficiency,

$$v_2 = \sqrt{\frac{1}{2 \varepsilon \sin \phi_2'}} \sqrt{2gh}. \quad \ldots\ldots\ldots\ldots\ldots \quad (51)$$

Other velocities corresponding to the maximum efficiency are

$$w_2 = \sqrt{\frac{\sin \phi_2' (3 + \sin \phi_2')}{2 (2 + \sin \phi_2')}} \sqrt{2gh}. \quad \ldots\ldots \quad (52)$$

$$c_2 = \frac{1-\sin\phi_2'}{\cos\phi_2} v_2$$

$$= \sqrt{\frac{(1-\sin\phi_2')(2-\sin^2\phi_2')}{2\sin\phi_2'(2+\sin\phi_2')}} \sqrt{2gh}. \quad \ldots \quad (53)$$

$$u_2 = w_2 \sin a_2 = c_2 \sin\phi_2$$

$$= \sqrt{\frac{\sin\phi_2'(1-\sin\phi_2')}{2(2+\sin\phi_2')}} \sqrt{2gh}. \quad \ldots\ldots\ldots \quad (54)$$

$$w_2 \cos a_2 = v_2 - c_2 \cos\phi_2 = v_2 \sin\phi_2'$$

$$= \sqrt{\frac{\sin\phi_2'(1+\sin\phi_2')}{(2+\sin\phi_2')}} \sqrt{2gh}. \quad \ldots \quad (55)$$

The direction of the absolute outflow velocity is given by

$$\tan a_2 = \frac{w_2 \sin a_2}{w_2 \cos a_2} = \frac{u_2}{w_2 \cos a_2}$$

$$= \sqrt{\frac{1-\sin\phi_2'}{2(1+\sin\phi_2')}} = \frac{1}{\sqrt{2}} \tan\frac{1}{2}\left(\frac{\pi}{2} - \phi_2'\right). \quad \ldots \quad (56)$$

Notice that the equation (49) can be written in the following simple form :—

$$\tan\phi_2' = \sqrt{2}\tan\phi_2. \quad \ldots\ldots\ldots\ldots\ldots \quad (49')$$

The accompanying Table II gives the maximum efficiency and the corresponding best velocities for several values of the discharge angle ϕ_2 from 15° to 90°. In this case also the efficiency increases with the diminution of the angle ϕ_2. The radial velocity is not so high as in Case 1; its value becomes greatest when $\sin\phi_2' = \sqrt{6}-2$ and $\tan^2\phi_2 = (3+\sqrt{24})^{-1}$, that is, when $\phi_2 = 19°35'$.

Table II. *Showing the Maximum Efficiency and the Corresponding Best Speeds for Case* 2.

Discharge angle ϕ_2	Maximum efficiency ε	Velocity of impeller, $v_2=\sqrt{2gh}\times$	Radial velocity, $u_2=\sqrt{2gh}\times$	Tangential velocity, $w_2 \cos \alpha_2 = \sqrt{2gh}\times$
90°	(·750)	·816	·000	·816
65°	·756	·834	·090	·792
45°	·775	·889	·163	·726
30°	·806	·990	·210	·626
20°	·843	1·139	·225	·521
15°	·869	1·275	·221	·451

Case 3. *A centrifugal pump with a vortex chamber, the spiral casing being not well-designed.*

In Case 3, the centrifugal pump is supposed to be provided with a vortex chamber extending from the outer surface of the impeller to another coaxial cylindrical surface of radius r_3 and bounded by two parallel plates, which are simply a continuation of the fixed casing for the impeller. The space thus formed for the vortex chamber is a plain disk-like space without any vanes in it; it is exactly the space $AABB$ shown in Fig. 1. The motion of water in this space is then a free spiral vortex.

Let w_3=the absolute outflow velocity of water from the vortex chamber,

r_3=the outer radius of the vortex chamber,

$m=r_2 \div r_3$=the ratio of the inner radius of the vortex chamber to the outer.

In this case a part of the absolute outflow velocity w_2 from the

impeller is converted into statical pressure in passing through the vortex chamber. By equation (8), this gain of pressure head is

$$(1-m^2)\frac{w_2^2}{2g}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (57)$$

Since the motion is a free spiral vortex, the direction of the outflow of water from the vortex chamber makes the same angle a_2 with the tangent to the circumference, but the magnitude of the outflow velocity is reduced; by equation (5), we have

$$w_3 = m\, w_2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (58)$$

The spiral casing being not well designed, the whole of w_3 will be very nearly wasted as in Case 1, and therefore

$$F = m^2\frac{w_2^2}{2g}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (59)$$

Substituting this in (35) and replacing w_2^2 by c_2 and v_2, we obtain

$$\varepsilon = 1-\frac{m^2(z^2+1-2z\cos\phi_2)}{2(1-z\cos\phi_2)}, \quad \ldots\ldots\ldots\ldots \quad (60)$$

in which

$$z = c_2 \div v_2.$$

Differentiating ε with respect to z and equating the result to zero, there is obtained a condition for maxmimum efficiency in the form of a relation between v_2 and c_2, thus:—

$$1-z\cos\phi_2 = \sin\phi_2, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (61)$$

i.e.,

$$v_2 - c_2\cos\phi_2 = v_2\sin\phi_2. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (61')$$

The corresponding maximum efficiency is easily found from (60):

$$\textit{Maximum}\ \varepsilon = 1-\frac{m^2\sin\phi_2}{1+\sin\phi_2}. \quad \ldots\ldots\ldots\ldots\ldots \quad (62)$$

The velocity of the outer circumference of the impeller for maximum efficiency is found to be

$$v_2=\sqrt{\frac{1}{2\,\varepsilon\sin\phi_2}}\sqrt{2gh}. \quad \ldots\ldots\ldots\ldots\ldots \quad (63)$$

Other velocities corresponding to the maximum efficiency are

$$w_2=\sqrt{\frac{\sin\phi_2}{\varepsilon\,(1+\sin\phi_2)}}\sqrt{2gh}, \quad \ldots\ldots\ldots\ldots \quad (64)$$

$$c_2=\frac{1-\sin\phi_2}{\cos\phi_2}v_2=v_2\tan\frac{1}{2}\left(\frac{\pi}{2}-\phi_2\right). \quad \ldots \quad (65)$$

$$u_2=w_2\sin\alpha_2=c_2\sin\phi_2=\tan\phi_2\,(1-\sin\phi_2)v_2$$

$$=\frac{1-\sin\phi_2}{\cos\phi_2}\sqrt{\frac{\sin\phi_2}{2\,\varepsilon}}\sqrt{2gh}. \quad \ldots\ldots\ldots\ldots \quad (66)$$

$$w_2\cos\alpha_2=v_2-c_2\cos\phi_2=v_2\sin\phi_2$$

$$=\sqrt{\frac{\sin\phi_2}{2\,\varepsilon}}\sqrt{2gh}. \quad \ldots\ldots\ldots\ldots \quad (67)$$

The direction of the absolute velocity of outflow from the impeller is given by

$$\tan\alpha_2=\frac{w_2\sin\alpha_2}{w_2\cos\alpha_2}=\frac{u_2}{w_2\cos\alpha_2}$$

$$=\frac{1-\sin\phi_2}{\cos\phi_2}=\tan\frac{1}{2}\left(\frac{\pi}{2}-\phi_2\right),$$

therefore $$\alpha_2=\frac{1}{2}\left(\frac{\pi}{2}-\phi_2\right). \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (68)$$

The accompanying Table III gives the maximum efficiency and the corresponding best velocities for several values of the discharge angle ϕ_2 from 15° to 90° and for values of $m=0{\cdot}9$, $0{\cdot}8$, and $0{\cdot}7$. It is seen from the table that by having a vortex chamber of large size a high efficiency can be obtained without much reducing the discharge angle ϕ_2. Thus when the diameter of the vortex chamber is 1·25 times the diameter of the impeller and the discharge angle ϕ_2 is 30°, the maximum efficiency is 79

per cent., which is greater by 12 per cent. than the efficiency of the otherwise same pump without a vortex chamber. For maximum efficiency, the radial velocity u_2 with which the water leaves the impeller ought to be rather high, unless the vortex chamber be very large. But a very high velocity of water through the pump means a large amount of the loss of head due to friction. Hence it may not be wise to run the pump at the speed of maximum efficiency, or in other words it is probably better to make the radial outflow velocity u_2 somewhat less than that corresponding to the maximum efficiency. The radial velocity u_2 for maximum efficiency varies with the discharge angle ϕ_2 and becomes greatest when

$$\sin \phi_2 = \frac{1}{1+\sqrt{2-m^2}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (69)$$

For $m=0{\cdot}9$, u_2 is maximum when $\phi_2=28°34'$; for $m=0{\cdot}8$, u_2 is maximum when $\phi_2=27°30'$; and for $m=0{\cdot}7$, u_2 is maximum when $\phi_2=26°39'$.

Table III. *Showing Maximum Efficiency and the Corresponding Best Velocities for Case 3.*

Discharge angle ϕ_2	Maximum efficiency ε	Velocity of impeller, $v_2=\sqrt{2gh}\times$	Radial velocity, $v_3=\sqrt{2gh}\times$	Tangential velocity, $v_2\cos\alpha_2=\sqrt{2gh}\times$
For $m=r_2\div r_3=0{\cdot}9$				
90°	(·595)	·917	·000	·917
65°	·615	1·016	·182	·819
45°	·664	1·031	·302	·730
30°	·730	1·170	·338	·585
20°	·794	1·357	·325	·464
15°	·834	1·523	·302	·394
For $m=r_2\div r_3=0{\cdot}8$				
90°	(·680)	·857	·000	·857
65°	·696	·891	·179	·807
45°	·735	·981	·287	·694
30°	·787	1·127	·325	·564
20°	·837	1·321	·317	·452
15°	·869	1·491	·296	·386
For $m=r_2\div r_3=0{\cdot}7$				
90°	.(755)	·814	·000	·814
65°	·767	·848	·170	·769
45°	·798	·941	·276	·666
30°	·837	1·093	·316	·547
20°	·875	1·293	·310	·442
15°	·899	1·466	·291	·379

Case 4. *A centrifugal pump with a vortex chamber, the spiral casing being well designed.*

In Case 4, the centrifugal pump is supposed to be provided with a vortex chamber as in Case 3 and the spiral casing to be so proportioned as to make the loss of head due to sudden change of velocity the least as in Case 2. The gain of pressure head in passing through the vortex chamber is exactly the same as in the last case, namely,

$$(1-m^2)\frac{w_2^2}{2g}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (70)$$

The radial component of the absolute outflow velocity from the vortex chamber is necessarily wasted. The spiral casing being so proportioned that the velocity of water in it is equal to one half of the tangential outflow velocity, that is,

$$w_4 = \frac{1}{2} w_3 \cos a_3$$

$$= \frac{1}{2} m\, w_2 \cos a_2, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (71)$$

the loss of head is, by (26′),

$$F = \frac{1}{2g}\left\{(w_3 \sin a_3)^2 + \frac{1}{2}(w_3 \cos a_3)^2\right\}$$

$$= \frac{m^2}{2g}\left\{(w_2 \sin a_2)^2 + \frac{1}{2}(w_2 \cos a_2)^2\right\}$$

$$= \frac{m^2}{2g}\left\{(c_2 \sin \phi_2)^2 + \frac{1}{2}(v_2 - c_2 \cos \phi_2)^2\right\}. \quad \ldots \quad (72)$$

In these expressions a_3 is the acute angle between the direction of w_3 and the tangent to the outer circumference of the vortex chamber. Since the motion in the vortex chamber is a free spiral vortex, the path of a particle of water makes a constant

angle with the radius vector and hence $a_3=a_2$. Substituting F in (35), the efficiency becomes

$$\varepsilon=1-\frac{m^2\{(c_2\sin\phi_2)^2+\frac{1}{2}(v_2-c_2\cos\phi_2)^2\}}{2\,v_2(v_2-c_2\cos\phi_2)}$$

$$=1-\frac{m^2\{(z\sin\phi_2)^2+\frac{1}{2}(1-z\cos\phi_2)^2\}}{2(1-z\cos\phi_2)}, \quad \ldots\ldots \quad (73)$$

in which $z=c_2\div v_2$. Differentiating this equation with respect to z and equating the result to zero, there is obtained the condition for maximum efficiency in the form of a relation between c_2 and v_2; thus:

$$1-z\cos\phi_2=\sin\phi_2', \quad \ldots\ldots \quad (74)$$

i.e.,
$$v_2-c_2\cos\phi_2=v_2\sin\phi_2', \quad \ldots\ldots \quad (74')$$

in which
$$\sin\phi_2'=\frac{\sin\phi_2}{\sqrt{\sin^2\phi_2+\frac{1}{2}\cos^2\phi_2}}. \quad \ldots\ldots \quad (75)$$

The value of the maximum efficiency then becomes

$$\textit{Maximum}\ \varepsilon=1-\frac{m^2\sin\phi_2'}{2(1+\sin\phi_2')}. \quad \ldots\ldots \quad (76)$$

The velocity of the outer circumference of the impeller for maximum efficiency then becomes

$$v_2=\sqrt{\frac{1}{2\,\varepsilon\sin\phi_2'}}\sqrt{2gh}. \quad \ldots\ldots \quad (77)$$

Other velocities corresponding to maximum efficiency are

$$w_2=v_2\sqrt{\frac{\sin^2\phi_2'(3+\sin\phi_2')}{2(1+\sin\phi_2')}}, \quad \ldots\ldots \quad (78)$$

$$c_2=v_2\sqrt{\frac{(1-\sin\phi_2')(2-\sin^2\phi_2')}{2(1+\sin\phi_2')}}, \quad \ldots\ldots \quad (79)$$

$$u_2=w_2\sin a_2=c_2\sin\phi_2$$

$$=v_2\sqrt{\frac{\sin^2\phi_2'(1-\sin\phi_2')}{2(1+\sin\phi_2')}}, \quad \ldots\ldots \quad (80)$$

$$w_2\cos a_2=v_2-c_2\cos\phi_2=v_2\sin\phi_2'. \quad \ldots\ldots \quad (81)$$

The direction of the absolute velocity of outflow from the impeller is given by

$$\tan \alpha_2 = \frac{w_2 \sin \alpha_2}{w_2 \cos \alpha_2} = \frac{u_2}{w_2 \cos \alpha_2}$$

$$= \sqrt{\frac{1-\sin \phi_2'}{2(1+\sin \phi_2')}} = \frac{1}{\sqrt{2}} \tan \frac{1}{2}\left(\frac{\pi}{2} - \phi_2'\right). \quad \ldots \quad (82)$$

Notice that the equation (75) can be written in the form

$$\tan \phi_2' = \sqrt{2} \tan \phi_2. \quad \ldots\ldots\ldots\ldots\ldots \quad (75')$$

The accompanying Table IV gives the maximum efficiency and the corresponding best velocities for several values of the discharge angle ϕ_2 from 15° to 90° and for values of $m=0{\cdot}9$, $0{\cdot}8$, and $0{\cdot}7$. The discharge angle ϕ_2 can be made very large without much loss of efficiency. It is a decided advantage to employ a large discharge angle ϕ_2, because the speed of rotation of the impeller as well as the radial velocity of water through the impeller will then become considerably reduced and in consequence the hydraulic friction and also the mechanical work wasted in shaft friction will become reduced. In Case 4 the radial velocity of water for maximum efficiency has a good practical value when the ratio m is $0{\cdot}9$ to $0{\cdot}7$ for all values of the discharge angle ϕ_2, say, from 60° to 15°. The radial outflow velocity $u_2 = w_2 \sin \alpha_2 = c_2 \sin \phi_2$ becomes greatest when

$$2 - 4 \sin \phi'_2 - (2-m^2) \sin^2 \phi'_2 = 0,$$

that is, when

$$\tan^2 \phi_2 = \frac{1}{(4-m^2) + \sqrt{8(4-m^2)}}. \quad \ldots\ldots\ldots \quad (83)$$

Thus for $m=0{\cdot}9$, u_2 is maximum when $\phi_2 = 19° 12'$,

„ $m=0{\cdot}8$, u_2 „ „ $\phi_2 = 18° 53'$,

and „ $m=0{\cdot}7$, u_2 „ „ $\phi_2 = 18° 37'$.

Table IV. *Showing Maximum Efficiency and the Corresponding Best Velocities for Case* 4.

Discharge angle ϕ_2	Sin ϕ_2'	Maximum efficiency ε	Velocity of impeller $v_2=\sqrt{2gh}\times$	Radial velocity $u_2=\sqrt{2gh}\times$	Tangential velocity $w_2\cos\alpha_2=\sqrt{2gh}\times$
For $m=r_2\div r_3=0{\cdot}9$					
90°	1·000	(·798)	·792	·000	·792
65°	·950	·803	·810	·087	·769
45°	·816	·818	·865	·157	·707
30°	·633	·843	·968	·206	·613
20°	·458	·873	1·119	·221	·512
15°	·354	·894	1·256	·217	·445
For $m=r_2\div r_3=0{\cdot}8$					
90°	1·000	(·840)	·772	·000	·772
65°	·950	·844	·790	·085	·750
60°	·926	·846	·804	·110	·744
45°	·816	·856	·846	·155	·691
30°	·633	·876	·950	·202	·601
20°	·458	·900	1·102	·218	·504
15°	·354	·916	1·241	·212	·440
For $m=r_2\div r_3=0{\cdot}7$					
90°	1·000	(·878)	·755	·000	·755
65°	·950	·881	·773	·083	·734
45°	·817	·890	·829	·152	·677
30°	·633	·905	·935	·198	·591
20°	·458	·923	1·087	·215	·498
15°	·354	·936	1·227	·212	·456

In centrifugal pumps with vortex chambers whose sides are parallel, the water before entering the spiral casing forms a free vortex and the velocity of whirl is inversely as the radius measured from the axis of the shaft. In order to avoid the waste of the kinetic energy of this velocity of whirl, the impeller must revolve in a large vortex chamber. Thus to obtain a theoretical efficiency of $\frac{3}{4}$, with a pump constructed with radial vanes, the diameter of the vortex chamber must be double the diameter of the impeller. To obtain the same efficiency with a pump having the discharge angle ϕ_2 equal to 30°, the diameter of the vortex chamber must be 1·63 times the diameter of the impeller. The higher is the efficiency of a centrifugal pump, the more cumbrous is the machine. *A pump of high efficiency and of a moderate size, however, can be obtained by a special construction given in the following Case 5.*

Case 5. *A centrifugal pump provided with a vortex chamber, which gradually diverges outwards, and in which there is a number of properly arranged guide vanes, the motion of water in the vortex chamber being forced vortex.*

The pump is here provided with a vortex chamber extending from the outer surface of the impeller to another co-axial cylindrical surface of radius r_3 and bounded by curved side plates diverging outwards, which are a continuation of the fixed casing for the pump, Fig. 6. The form of the vortex chamber is that given in Fig. 2. The radial velocity u and the resultant velocity w at any point P, Fig. 2, in the vortex chamber have the following relation :—

$$r\,b\,u = r_2 b_2 u_2 = r_3 b_3 u_3,$$

$$u = w \sin \alpha, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (19)$$

$$b\,r\,w\sin a = b_2 r_2 w_2 \sin a_2 = b_3 r_3 w_3 \sin a_3. \quad \ldots\ldots \quad (20)$$

Hence by making b_3 greater than b_2 and a_3 greater than a_2, it is possible to reduce to a large extent the velocity of outflow from

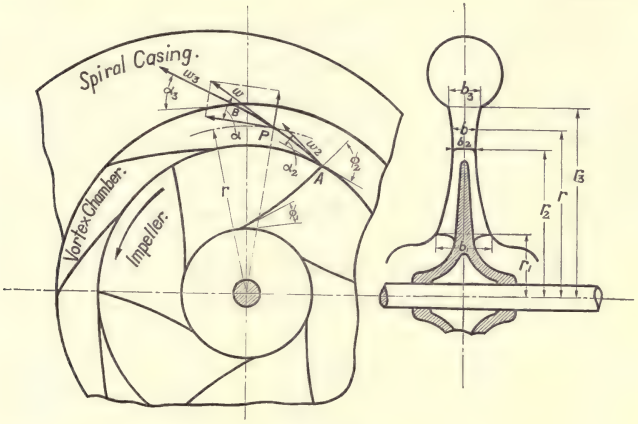


FIG. 6.

the vortex chamber, of which the diameter is only moderately large. Both the radial and the resultant velocity can be made to decrease gradually by giving suitable values to a and b, the angle a being determined by the directions of fixed guide vanes, a number of which is placed in the vortex chamber. The motion of water through such a vortex chamber is then a constrained or forced vortex.

Putting $$m = \frac{b_2 r_2 \sin a_2}{b_3 r_3 \sin a_3}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (84)$$

the equation (20) gives

$$w_3 = m\,w_2,$$

which is exactly of the same form as the equation (58). Hence in a centrifugal pump belonging to Case 5, if no special atten-

tion is paid to the proportions and construction of the spiral casing, it follows at once that all the reasoning in Case 3 and the formulæ from (57) to (69) are applicable to Case 5 without any alteration.

The accompanying Table V gives the maximum efficiency and the corresponding best velocities for several values of discharge angle ϕ_2 and for small values of m=·6, ·5, ·4, ·3, and ·2 suitable for pumps belonging to Case 5. Excepting that the quantity m has now an extended meaning given by (84), Table V is simply a continuation of Table III for low values of m.

Table V. *Showing Maximum Efficiency and the Corresponding Best Velocities for Case 5.*

Discharge angle ϕ_2	Maximum efficiency ε	Velocity of impeller $v_2=\sqrt{2gh}\times$	Radial velocity $u_2=\sqrt{2gh}\times$	Tangential velocity $w_2\cos\alpha_2=\sqrt{2gh}\times$
For $m=\frac{b_2 r_2 \sin\alpha_2}{b_3 r_3 \sin\alpha_3}=0{\cdot}6$				
90°	(·820)	·781	·000	·781
65°	·833	·814	·167	·738
45°	·851	·912	·267	·645
30°	·880	1·066	·308	·533
20°	·908	1·269	·304	·434
15°	·926	1·444	·280	·374
For $m=\frac{r_2 b_2 \sin\alpha_2}{r_3 b_3 \sin\alpha_3}=0{\cdot}5$				
90°	(·875)	·756	·000	·756
65°	·881	·791	·159	·717
45°	·896	·888	·260	·628
30°	·917	1·044	·302	·522

Table V (*continued.*)

Discharge angle ϕ_2	Maximum efficiency ε	Velocity of impeller $v_2=\sqrt{2gh}\times$	Radial velocity $u_2=\sqrt{2gh}\times$	Tangential velocity $w_2 \cos \alpha_2 = \sqrt{2gh}\times$
For $m=\frac{r_2 b_2 \sin \alpha_2}{r_3 b_3 \sin \alpha_3}=0{\cdot}4$				
90°	(·920)	·737	·000	·737
65°	·924	·773	·155	·700
45°	·934	·870	·255	·615
30°	·947	1·038	·300	·519
For $m=\frac{r_2 b_2 \sin \alpha_2}{r_3 b_3 \sin \alpha_3}=0{\cdot}3$				
90°	(·955)	·724	·000	·724
65°	·957	·759	·153	·688
45°	·963	·857	·251	·606
30°	·970	1·015	·293	·508
For $m=\frac{r_2 b_2 \sin \alpha_2}{r_3 b_3 \sin \alpha_3}=0{\cdot}2$				
90°	(·980)	·714	·000	·714
65°	·980	·750	·151	·725
45°	·983	·848	·248	·600
30°	·987	1·007	·291	·503

In a centrifugal pump having a vortex chamber of an ordinary form consisting of simple disk-like space bounded by parallel walls without guide vanes, it would require an unmanageably large vortex chamber to reduce the ratio $m=r_2 \div r_3$ to so low a value as 0·3 or 0·2. In a free vortex pump, therefore, the only

practical way of obtaining a high efficiency is by making the discharge angle ϕ_2 small. But a very small value of this angle ϕ_2 necessitates a high peripheral speed of the impeller, causing a large amount of loss of work in friction. For example in a free vortex pump of Case 3, let $m = r_2 \div r_3 = 0{\cdot}75$ and $\phi_2 = 25°$, then the theoretical maximum efficiency is 0·83 and the corresponding peripheral velocity of the impeller is $1{\cdot}21\sqrt{2gh}$. Now in a forced vortex pump it is quite easy to reduce the ratio m to 0·2 or to a still smaller value, thereby securing a very high efficiency; and at the same time the discharge angle ϕ_2 can be made very large without an appreciable loss of efficiency. A very important practical advantage of making ϕ_2 large is the lowness of the peripheral speed of the impeller. In a forced vortex pump made at Shibaura Engineering Works, Tokyo, according to the design of the author, and tested against lifts up to 130 ft., the principal dimensions were as follows :—

Lift for which designed	120 ft.
Delivery per minute	112 cub. ft.
Dia. of suction and delivery	7″
Outer dia. of impeller, $2r_2$	$14\frac{1}{4}$″
Inner dia. of impeller, $2r_1$	$7\frac{1}{8}$″
Dia. of vortex chamber, $2r_3$	20″
Axial width of impeller at outlet, b_2... ...	$\frac{3}{4}$″
Axial width of vortex chamber at outlet, b_3...	$1\frac{1}{8}$″
Discharge angle of impeller vanes, ϕ_2 ...	75°
Entrance angle of guide vanes, α_2	7°31′
Discharge angle of guide vanes, α_3	31°48′

In this pump, the ratio m has the following value :—

$$m=\frac{b_2 r_2 \sin a_2}{b_3 r_3 \sin a_3}=\frac{\frac{3}{4}\times 7\frac{1}{8}\times \sin 7°31'}{1\frac{1}{8}\times 10\times \sin 31°48'}=0{\cdot}118,$$

and neglecting friction altogether, the maximum efficiency is 0·99, the corresponding peripheral speed of the impeller being only $0{\cdot}72\sqrt{2gh}$ and the radial velocity of outflow from the impeller $0{\cdot}092\sqrt{2gh}$. The results of tests of this pump carried out at Shibaura Engineering Works will be given in another Paper *en suite*.

Speed at which Delivery Commences. When a centrifugal pump is started, the water filling the space between the outer and inner surfaces of the impeller will, in the first instance, rotate as a solid mass with the impeller and the discharge will not commence till the speed of the outer circumference of the impeller reaches the value given by

$$v_2^2\left(1-\frac{r_1^2}{r_2^2}\right)=2gh. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (85)$$

Hence it is often difficult to start a centrifugal pump in which the outer radius is not much greater than the inner. If, for example, the outer radius of the impeller is twice that of the inner and if the suction and delivery pipes are both *filled with water up to the working lift h before starting*, there will be no delivery until the speed of the outer periphery reaches the value

$$v_2=\left\{1-(\tfrac{1}{2})^2\right\}^{-\frac{1}{2}}\sqrt{2gh}$$

$$=1{\cdot}15\sqrt{2gh}$$

When the pump is once started, this initial speed may be gradually reduced without stopping the delivery. Supposing that the

pump is provided with a vortex chamber whose dia. is $1\frac{1}{4}$ times that of the impeller and that the discharge angle ϕ_2 is 65°, the speed would have to be reduced to $\cdot 89\sqrt{2gh}$ for maximum efficiency. The delivery would cease entirely when the speed is still more reduced, down to $\cdot 86\sqrt{2gh}$. For the coefficients of these two velocities see Table III. The equation (85) given above may be obtained as follows:—When there is no delivery of water, the equations (29) and (30) evidently become

$$\frac{p_1}{\rho} = h_a - h_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (29')$$

and

$$\frac{p_2}{\rho} = h_a + h_2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (30')$$

Now neglecting the disk friction, the equation (31) becomes

$$\frac{p_2}{\rho} - \frac{p_1}{\rho} = \frac{v_2^2}{2g} - \frac{v_1^2}{2g}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (31')$$

Since $\frac{v_1}{r_1} = \frac{v_2}{r_2}$, the elimination of p_1 and p_2 from (31′) by means of (29′) and (30′) gives a resulting equation which is (85).

Efficiency and Delivery at a Speed Differing from That of Maximum Efficiency. It has been clearly shown that in any centrifugal pump working against a given lift, there exists some definite speed of the impeller for which the efficiency is maximum. The speed of the impeller for the maximum efficiency and the corresponding principal velocities of the water passing through the pump have also been fully considered. Now supposing that a pump is run at a speed differing from that of the maximum efficiency, questions arise how far will the efficiency fall short of that maximum value and in what proportion will the delivery of water vary in consequence? These two points and also another point of importance, namely, the effect of the

variation of the discharge angle ϕ_2 on the efficiency and delivery, will now be considered.

For this purpose a general expression for the total loss of head F can be obtained in the following form :—

$$F = \frac{m^2}{2g}\left\{u_2^2 + i(v_2 - c_2 \cos \phi_2)^2\right\} \quad \ldots\ldots\ldots \quad (86)$$

$$= \frac{m^2}{2g}\left\{u_2^2 + i(v_2 - u_2 \cot \phi_2)^2\right\} \quad \ldots\ldots\ldots \quad (86')$$

Making m and i in this equation both equal to unity, the result is the equation (36) of Case 1 ; making m equal to unity and i equal to $\frac{1}{2}$, the result is the equation (46) of Case 2 ; making i equal to unity, the result is the equation (59) of Case 3 and Case 5 ; and making i equal to $\frac{1}{2}$, the result is the equation (72) of Case 4. The above expression for F is, therefore, applicable to any of the cases previously considered, by assigning proper values to m and i. The quantity i has been stated to be either a unity or else a half. Strictly speaking, however, it may be any number greater than a half up to a unity in Cases 1, 3, and 5. For, taking Case 1 and referring to the equation (26), the loss of head is more accurately

$$F = \frac{1}{2g}\left\{(w_2 \sin \alpha_2)^2 + (w_2 \cos \alpha_2 - w_4)^2 + w_4^2\right\},$$

in which w_4 is the velocity of water in the spiral casing. Writing $w_4 = j\,(w_2 \cos \alpha_2)$ where j is supposed to have any value from zero to unity except one half, we have

$$F = \frac{1}{2g}\left\{(w_2 \sin \alpha_2)^2 + [(1-j)^2 + j^2](w_2 \cos \alpha_2)^2\right\}$$

$$= \frac{1}{2g}\left\{u_2^2 + i(v_2 - u_2 \cot \phi_2)^2\right\},$$

which is the equation (86) and in which i stands for $(1-j)^2+j^2$. Hence the quantity i is always greater than a half and may reach a unity. Substituting the general value of F' above given in the equation (34), we have

$$h+\frac{m^2}{2g}\left\{u_2^2+i(v_2-u_2\cot\phi_2)^2\right\}$$
$$=\frac{1}{g}v_2(v_2-u_2\cot\phi_2),$$

i.e.,
$$m^2u_2^2(1+i\cot^2\phi_2)+2u_2v_2(1-im^2)\cot\phi_2$$
$$=(2-im^2)v_2^2-2gh. \quad \ldots\ldots \quad (87)$$

Let
$$\cot\phi_2=\frac{1}{\sqrt{i}}\cot\phi_2'.$$

Then
$$m^2u_2^2+2u_2v_2(1-im^2)\frac{1}{\sqrt{i}}\sin\phi_2'\cos\phi_2'$$
$$=\left\{(2-im^2)v_2^2-2gh\right\}\sin^2\phi_2'. \quad \ldots\ldots \quad (87')$$

This equation shows that in a centrifugal pump, whatever may be the discharge angle ϕ_2, the delivery of water will cease entirely when the speed of the impeller is reduced down to

$$v_2=\sqrt{\frac{2gh}{2-im^2}}. \quad \ldots\ldots \quad (88)$$

Solving the equation (87′) for u_2, we obtain

$$u_2=\frac{q\sin\phi_2'}{m^2\sqrt{i}}\sqrt{2gh}, \quad \ldots\ldots \quad (89)$$

$$=k\sqrt{2gh}, \quad \ldots\ldots \quad (89A)$$

in which
$$q=\sqrt{\left\{1-(1-im^2)^2\sin^2\phi_2'\right\}x^2-im^2}$$
$$-(1-im^2)x\cos\phi_2', \quad \ldots\ldots \quad (90)$$

$$\tan \phi_2' = \frac{1}{\sqrt{i}} \tan \phi_2, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (91)$$

and $$v_2 = x\sqrt{2gh}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (92)$$

Now by equation (35), the efficiency of the pump is

$$\varepsilon = \frac{2gh}{2\, v_2(v_2 - u_2 \cot \phi_2)}$$

$$= \frac{2gh}{2\, x\sqrt{2gh}\left(x\sqrt{2gh} - \frac{q \sin \phi_2'}{m^2\sqrt{i}}\sqrt{2gh}\frac{1}{\sqrt{i}}\cot \phi_2'\right)}$$

$$= \frac{1}{2\, x\left(x - \frac{q \cos \phi_2'}{i\, m^2}\right)}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (93)$$

This equation enables us to calculate the efficiency of a centrifugal pump when it is driven at any speed against a given lift; and since the delivery of water is $Q = u_2 A_2$ where A_2 is a constant for a given pump and equal to $2\pi r_2 b_2 =$ the area of the cylindrical outlet surface of the impeller, the radial velocity u_2 may be taken as a measure of the delivery of water and therefore the equation (89) enables us to make an estimate of the delivery by a given pump when it is driven at a given speed against a given lift. Consider the several cases separately and more in detail.

Case I. In this case $m = 1$ and $i = 1$, so that $\phi_2' = \phi_2$ and $q = \sqrt{x^2 - 1}$. Hence (89) and (93) take the following forms:—

$$u_2 = \sin \phi_2 \sqrt{x^2 - 1}\sqrt{2gh} \quad \ldots\ldots\ldots\ldots\ldots \quad (89')$$

and $$\varepsilon = \frac{1}{2x(x - \cos \phi_2 \sqrt{x^2 - 1})}. \quad \ldots\ldots\ldots\ldots \quad (93')$$

In the accompanying Table VI are given the values of u_2 and ε calculated by these formulæ for several values of x and for a few values of ϕ_2. The speed of no delivery of water is, in this case, equal to the velocity due to the lift, as is shown by (88).

Table VI. *Showing the Efficiency and the Radial Velocity for a Given Speed of the Impeller for Case* 1.

Discharge angle ϕ_2	Value of x in $v_2=x\sqrt{2gh}$	Efficiency ε	Value of k in $u_2=k\sqrt{2gh}$
90°	1·00	(·500)	·000
	1·05	·454	·320
	1·10	·413	·458
	1·15	·378	·568
	1·20	·347	·663
60°	1·00	(·500)	·000
	1·05	·535	·277
	1·10	·522	·397
	1·15	·501	·470
	1·20	·480	·574
30°	1·00	(·500)	·000
	1·05	·616	·160
	1·10	·646	·229
	1·15	·661	·284
	1·20	·666	·332
	1·3	·663	·415
	1·4	·648	·490

Case 2. In this case $m=1$ and $i=\frac{1}{2}$, so that $\tan\phi_2' = \sqrt{2}\tan\phi_2$ and

$$2q=\sqrt{(3+\cos^2\phi_2')x^2-2}-x\cos\phi_2'.$$

Hence
$$u_2=\frac{2q\sin\phi_2'}{\sqrt{2}}\sqrt{2gh} \quad \ldots\ldots\ldots\ldots\ldots \quad (89^2)$$

and $$\varepsilon = \frac{1}{2x(x - 2q\cos\phi_2')}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (93^2)$$

In the accompanying Table VII are given the values of u_2 and ε calculated by these formulæ for several values of x and for a few values of ϕ_2. The speed of the impeller at which the delivery of water ceases entirely is found by (88) and is equal to the velocity due to $\frac{2}{3}$ of the lift, that is, $0{\cdot}816\sqrt{2gh}$.

Table VII. *Showing the Efficiency and the Radial Velocity for a Given Speed of the Impeller for Case* 2.

Discharge angle ϕ_2	Value of x in $v_2 = x\sqrt{2gh}$	Efficiency ε	Value of k in $u_2 = k\sqrt{2gh}$
90°	·816	(·750)	·000
	·85	·692	·289
	·90	·560	·592
	1·00	·500	·707
60°	·816	(·750)	·000
	·85	·759	·130
	·90	·741	·365
	1·00	·677	·428
30°	·816	(·750)	·000
	·90	·780	·116
	·95	·806	·171
	1·0	·806	·219
	1·05	·802	·264
	1·1	·796	·305
	1·15	·796	·349
	1·2	·775	·381
	1·3	·747	·453

Case 3. In this case $i=1$, so that $\phi_2'=\phi_2$ and

$$q=\sqrt{\left\{1-(1-m^2)^2\sin^2\phi_2\right\}x^2-m^2}-(1-m^2)x\cos\phi_2. \quad \ldots \quad (90^3)$$

Hence
$$u_2=\frac{q\sin\phi_2}{m^2}\sqrt{2gh} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (89^3)$$

and
$$\varepsilon=\frac{1}{2x\left(x-\frac{q\cos\phi_2}{m^2}\right)}. \quad \ldots\ldots\ldots\ldots\ldots \quad (93^3)$$

By equation (88), the velocity of the outer periphery of the impeller for no delivery of water is

$$v_2=\sqrt{\frac{2gh}{2-m^2}},$$

for any value of the discharge angle ϕ_2. Now if the diameter of the vortex chamber be so large that the quantity m can be neglected without sensible error, the velocity of the impeller for no delivery becomes

$$v_2=\sqrt{gh}$$

Thus by driving the impeller at a speed equal to the velocity due to *one half of the lift* it is possible to keep the level of water in the delivery pipe at a constant height *equal to the lift*, without discharging any water. The accompanying three Tables VIIIA, VIIIB, VIIIC give the values of u_2 and ε calculated by the formulæ (89^3) and (93^3) for several values of x and for a few values of ϕ_2, when $m=0{\cdot}9$, $0{\cdot}8$, and $0{\cdot}7$.

Table VIII A. *Showing the Efficiency and the Radial Velocity for a Given Speed of the Impeller for Case* 3, $m=0{\cdot}9$.

Discharge angle ϕ_2	Value of x in $v_2=x\sqrt{2gh}$	Efficiency ε	Value of k in $u_2=k\sqrt{2gh}$
90°	·917	(·595)	·000
	·95	·554	·302
	1·00	·500	·482
	1·05	·454	·621
60°	·917	(·595)	·000
	·95	·610	·182
	1·00	·618	·330
	1·05	·599	·442
30°	·917	(·595)	·000
	·95	·651	·082
	1·00	·683	·161
	1·05	·713	·221
	1·10	·724	·273
	1·15	·729	·319
	1·20	·729	·363

Table VIII B. *Showing the Efficiency and the Radial Velocity for a Given Speed of the Impeller for Case* 3, $m=0{\cdot}8$.

Discharge angle ϕ_2	Value of x in $v_2=x\sqrt{2gh}$	Efficiency ε	Value of k in $u_2=k\sqrt{2gh}$
90°	·858	(·680)	·000
	·90	·617	·398
	·95	·554	·423
	1·00	·500	·750
60°	·858	(·680)	·000
	·90	·703	·190
	·95	·695	·334
	1·00	·681	·450
30°	·858	(·680)	·000
	·90	·724	·077
	·95	·754	·146
	1·00	·772	·228
	1·05	·782	·255
	1·10	·785	·301
	1·20	·766	·351

Table VIII C. *Showing the Efficiency and the Radial Velocity for a Given Speed of the Impeller for Case* 3, $m=0{\cdot}7$.

Discharge angle ϕ_2	Value of x in $v_2=x\sqrt{2gh}$	Efficiency ε	Value of k in $u_2=k\sqrt{2gh}$
90°	·814	(·755)	·000
	·85	·692	·430
	·90	·617	·675
	·95	·554	·860
60°	·814	(·755)	·000
	·85	·772	·152
	·90	·768	·306
	·95	·765	·454
30°	·814	(·755)	·000
	·85	·782	·056
	·90	·807	·122
	·95	·822	·179
	1·00	·831	·230
	1·05	·835	·277
	1·10	·838	·322
	1·20	·832	·404

Case 4. In this case $i=\frac{1}{2}$ so that $\tan\phi_2'=\sqrt{2}\tan\phi_2$ and

$$q=\sqrt{\left\{1-(1-\tfrac{1}{2}m^2)^2\sin^2\phi_2'\right\}x^2-\tfrac{1}{2}m^2}-(1-\tfrac{1}{2}m^2)x\cos\phi_2'.$$

Hence
$$u_2=\frac{\sqrt{2}\,q\sin\phi_2'}{m^2}\sqrt{2gh}, \quad \ldots\ldots\ldots\ldots \quad (89^4)$$

and
$$\varepsilon=\frac{1}{2x\left(x-\dfrac{2q\cos\phi_2'}{m^2}\right)}. \quad \ldots\ldots\ldots\ldots \quad (93^4)$$

By equation (88), the velocity of the outer periphery of the impeller for no delivery of water is

$$v_2=\sqrt{\frac{2gh}{2-\frac{1}{2}m^2}}.$$

The accompanying three Tables IXA, IXB, IXC give the values of u_2 and ε calculated by (89^4) and (93^4) for several values of x and for a few values of ϕ_2 when $m=0{\cdot}9$, $0{\cdot}8$, and $0{\cdot}7$.

Table IX A. *Showing the Efficiency and the Radial Velocity for a Given Speed of the Impeller for Case* 4, $m=0{\cdot}9$.

Discharge angle ϕ_2	Value of x in $v_2=x\sqrt{2gh}$	Efficiency ε	Value of k in $u_2=k\sqrt{2gh}$
90°	·792	(·798)	·000
	·85	·692	·434
	·90	·617	·600
60°	·792	(·798)	·000
	·85	·799	·198
	·90	·775	·318
	·95	·743	·419
30°	·792	(·798)	·000
	·85	·830	·081
	·90	·838	·137
	·95	·843	·188
	1·0	·842	·234
	1·1	·831	·319
	1·2	·810	·396

Table IX B. *Showing the Efficiency and the Radial Velocity for a Given Speed of the Impeller for Case* 4, $m=0{\cdot}8$.

Discharge angle ϕ_2	Value of x in $v_2=x\sqrt{2gh}$	Efficiency ε	Value of k in $u_2=k\sqrt{2gh}$
90°	·772	(·840)	·000
	·80	·781	·342
	·85	·692	·578
60°	·772	(·840)	·000
	·80	·858	·124
	·85	·832	·248
	·90	·807	·368
30°	·772	(·840)	·0000
	·80	·853	·0385
	·85	·868	·0992
	·90	·875	·153
	·95	·876	·202
	1·00	·875	·248
	1·10	·864	·331
	1·20	·845	·408

Table IX C. *Showing the Efficiency and the Radial Velocity for a Given Speed of the Impeller for Case* **4**, $m=0{\cdot}7$.

Discharge angle ϕ_2	Value of x in $v_2=x\sqrt{2gh}$	Efficiency ε	Value of k in $u_2=k\sqrt{2gh}$
90°	·755	(·878)	·000
	·80	·781	·501
	·85	·692	·740
60°	·755	(·878)	·000
	·80	·881	·157
	·85	·865	·295
	·90	·838	·410
30°	·755	(·878)	·0000
	·80	·891	·0569
	·85	·900	·113
	·90	·904	·165
	·95	·905	·213
	1·00	·903	·257
	1·10	·892	·341
	1·20	·876	·418

Comparison and Discussion. Numerical values of efficiency and two other quantities given in the foregoing eight Tables from VI to IX C have been re-arranged and formed into six other Tables from X to XV so as to facilitate comparison. Going over these six tables, the exact dependence upon each other of the three quantities therein given can be very clearly traced out.

1°. Let us first confine our attention to pumps having the discharge angle ϕ_2 equal to 30°. Referring to Table X it is seen that the velocity of the impeller for maximum efficiency ranges from $1{\cdot}2\sqrt{2gh}$ to $0{\cdot}95\sqrt{2gh}$, being highest in Case 1 and lowest in Case 4. A deviation of the velocity of the impeller by $0{\cdot}10\sqrt{2gh}$ from the best velocity has the effect of lowering the efficiency by about two per cent. in Case 1 and by a much smaller percentage in most other cases; a greater deviation from the best velocity causes the loss of efficiency in a somewhat faster proportion especially in Case 1. As has been stated before, the radial velocity u_2 may be taken as a measure of the delivery of water by a given pump when working against a given lift. Referring to Table XI it is seen that in Cases 1 and 3, this radial velocity u_2 for maximum efficiency is rather high, being $0{\cdot}30\sqrt{2gh}$ to $0{\cdot}33\sqrt{2gh}$. High radial velocities such as these would cause a great deal of loss of head due to hydraulic friction, unless the lift h be very small, say, up to 8 ft. or 9 ft. only. In Cases 2 and 4, the radial velocity u_2 for maximum efficiency is from $0{\cdot}19\sqrt{2gh}$ to $0{\cdot}22\sqrt{2gh}$, which may be regarded as moderate for moderate lifts, but probably not low enough for lifts higher than about 18 ft. or 20 ft.

2°. Secondly consider pumps having the discharge angle ϕ_2 equal to 60°. Referring to Table XII it is seen that the speed

of the impeller for maximum efficiency ranges from $0{\cdot}80\sqrt{2gh}$ to $1{\cdot}05\sqrt{2gh}$, being highest in Case 1 and lowest in Case 4. If a pump belonging to any of the cases and having the discharge angle $\phi_2=30°$ is compared with one belonging to the same case and having $\phi_2=60°$, the velocity v_2 of the maximum efficiency of the first pump is higher than that of the second by $0{\cdot}15\sqrt{2gh}$ or more. If a pump belonging to Case 1 and having $\phi_2=30°$ is compared with one belonging to Case 4 and having $\phi_2=60°$, the velocity v_2 of the first pump is $1\frac{1}{2}$ times that of the second, and therefore if these two pumps are driven at the same peripheral velocity and both developed their power at their maximum efficiency, the lift of water by the second pump will be $2\frac{1}{4}$ times that by the first, the efficiency being 0·88 and 0·67 respectively. Thus there is a gain in lift of 125 per cent. and a gain in efficiency of 21 per cent. both in favour of the second pump. These results will of course be modified by hydraulic friction to a greater or less extent according to the velocity of water through the pump. The effect of hydraulic friction will be dealt with further on.

3°. Thirdly consider pumps having the discharge angle ϕ_2 equal to 90°. When the vanes are radial, whatever may be the forms of the pump in other repects, the efficiency is the same for the same velocity of the impeller and increases continuously as the velocity of impeller decreases down to the limiting velocity at which the discharge of water ceases entirely, so that there does not exist a velocity of maximum efficiency. The velocity of the impeller for no delivery of water for the eight forms of the pump given in the foregoing tables ranges from $0{\cdot}76\sqrt{2gh}$ to $1{\cdot}00\sqrt{2gh}$ and a small rise of the velocity above these values produces a considerable radial velocity v_2 and consequently the

delivery. In general, when the discharge angle ϕ_2 is equal to or slightly less than 90°, a small variation of the velocity of the pump causes a comparatively large variation of the delivery, so that pumps of this type will be somewhat more liable to become unsteady as regards the quantity of delivery than pumps having smaller discharge angle. The larger is the discharge angle ϕ_2, the greater becomes the loss of efficiency of the pump; but on the other hand in vortex chamber pumps there is a practical advantage in making ϕ_2 moderately large especially for high lifts, because then the speed of the impeller and therefore of the pump shaft can be reduced to a very large extent and the loss due to mechanical friction accordingly diminished.

Table X. *Showing the Efficiency for Successive Velocities of the Impeller when* $\phi_2=30°$.

Value of x in $v_2=x\sqrt{2gh}$	Case 1	Case 3			Case 2	Case 4		
		$m=0·9$	$m=0·8$	$m=0·7$		$m=0·9$	$m=0·8$	$m=0·7$
0·80							·853	·891
0·85				·782		·830	·868	·900
0·90			·724	·807	·780	·838	·875	·904
0·95		·651	·754	·822	·806	**·843**	**·876**	**·905**
1·00		·683	·772	·831	**·806**	·842	·875	·903
1·05	·616	·713	·782	·835	·802			
1·10	·646	·724	**·785**	**·838**	·796	·831	·864	·892
1·15	·661	**·729**			·796			
1·2	**·666**	·729	·766	·832	·775	·810	·845	·876
1·3	·663				·747			
1·4	·648							

Table XI. *Showing the Coefficient k in the Expression for the Radial Velocity $u_2 = k\sqrt{2gh}$, for Successive Velocities of the Impeller, when $\phi_2 = 30°$.*

Value of x in $v_2 = x\sqrt{2gh}$	Case 1	Case 3			Case 2	Case 4		
		$m=0{\cdot}9$	$m=0{\cdot}8$	$m=0{\cdot}7$		$m=0{\cdot}9$	$m=0{\cdot}8$	$m=0{\cdot}7$
0·80							·039	·057
0·85				·056		·081	·099	·113
0·90			·077	·122	·116	·137	·153	·165
0·95		·082	·146	·179	·171	**·188**	**·202**	**·213**
1·00		·161	·228	·230	**·219**	·234	·248	·257
1·05	·160	·221	·255	·277	·264			
1·10	·229	·273	**·301**	**·322**	·306	·319	·331	·341
1·15	·284	**·319**			·349			
1·20	**·332**	·363	·351	·404	·381	·396	·408	·418
1·3	·415				·453			
1·4	·490							

[TABLE.

Table XII. *Showing the Efficiency* ε *for Successive Velocities of the Impeller, when* $\phi_2 = 60°$.

Value of x in $v_2 = x\sqrt{2gh}$	Case 1	Case 3			Case 2	Case 4		
		$m=0{\cdot}9$	$m=0{\cdot}8$	$m=0{\cdot}7$		$m=0{\cdot}9$	$m=0{\cdot}8$	$m=0{\cdot}7$
0·80							**·858**	**·881**
0·85				**·772**	**·759**	**·799**	·832	·865
0·90		·610	**·703**	·768	·741	·775	·807	·838
0·95		**·618**	·695	·765		·743		
1·00	(·500)	·599	·681		·677			
1·05	**·535**							
1·10	·522							
1·15	·501							

Table XIII. *Showing the Coefficient* k *in the Expression for the Radial Velocity* $u_2 = k\sqrt{2gh}$, *for Successive Velocities of the Impeller, when* $\phi_2 = 60°$.

Value of x in $v_2 = x\sqrt{2gh}$	Case 1	Case 3			Case 2	Case 4		
		$m=0{\cdot}9$	$m=0{\cdot}8$	$m=0{\cdot}7$		$m=0{\cdot}9$	$m=0{\cdot}8$	$m=0{\cdot}7$
0·80							**·124**	**·157**
0·85				**·152**	**·130**	**·198**	·248	·295
0·90			**·190**	·306	·365	·318	·368	·410
0·95		**·182**	·334	·454		·419		
1·00	·000	·330	·450		·428			
1·05	**·277**	·442						
1·10	·397							
1·15	·470							

Table XIV. *Showing the Efficiency ε for Successive Velocities of the Impeller, when $\phi_2 = 90°$.*

Value of x in $v_2 = x\sqrt{2gh}$	Case 1	Case 3			Case 2	Case 4		
		$m=0{\cdot}9$	$m=0{\cdot}8$	$m=0{\cdot}7$		$m=0{\cdot}9$	$m=0{\cdot}8$	$m=0{\cdot}7$
0·80							·781	·781
0·85				·692	·692	·692	·692	·692
0·90			·617	·617	·617	·617	·617	
0·95		·554	·554	·554	·554			
1·00	(·500)	·500	·500		·500			
1·05	·454	·454						
1·10	·413							
1·15	·378							

Table XV. *Showing the Coefficient k in the Expression for the Radial Velocity $u_2 = k\sqrt{2gh}$, for Successive Velocities of the Impeller, when $\phi_2 = 90°$.*

Value of x in $v_2 = x\sqrt{2gh}$	Case 1	Case 3			Case 2	Case 4		
		$m=0{\cdot}9$	$m=0{\cdot}8$	$m=0{\cdot}7$		$m=0{\cdot}9$	$m=0{\cdot}8$	$m=0{\cdot}7$
0·80							·342	·501
0·85				·430	·289	·434	·578	·740
0·90			·398	·675	·592	·600		
0·95		·302	·423	·860				
1·00	·000	·482	·750		·707			
1·05	·320	·621						
1·10	·458							
1·15	·568							

Efficiency and Velocity of Forced Vortex Pump. Numerical values of theoretical efficiency and of radial velocity of water for forced vortex pump of Case 5 have been calculated by means of equations (90^3), (89^3), and (93^3), and given in Tables XVI and XVII. It is seen from these tables that for small values of the ratio,

$$m = \frac{b_2 r_2 \sin \alpha_2}{b_3 r_3 \sin \alpha_3}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (84)$$

say, not greater than about $\frac{1}{2}$, the efficiency of the pump remains practically the same for all discharge angle ϕ_2 from about 45° to 75° and that for any given angle ϕ_2 the efficiency does not much vary with the velocity of the impeller. But an increased velocity of the impeller causes a corresponding increase of the radial velocity, which increase is greater the larger is the angle ϕ_2, as can be seen from Table XVII. Thus a pump having the discharge angle ϕ_2 equal to or slightly less than 90° is unsteady as regards the quantity of delivery per unit time, or in other words, a small variation of the velocity of impeller produces a comparatively large variation of the delivery. Again a forced vortex pump, in which m is less than ·6 and ϕ_2 is 20° to 30°, requires a peripheral velocity of the impeller of about $1\frac{1}{4}\sqrt{2gh}$ to $\sqrt{2gh}$; while a similar pump having ϕ_2 equal to from 45° to 75° requires a peripheral velocity of about $\cdot 90\sqrt{2gh}$ to $\cdot 73\sqrt{2gh}$. Now by properly arranging a number of guide vanes in the vortex chamber, it is quite easy to make the ratio m less than ·3 and to keep the angle ϕ_2 large, say, from 45° to 75°. Theoretical efficiency higher than 95 per cent. can then be obtained at a velocity of the impeller lower than that due to the lift against which the pump works. These conclusions will of course be

modified in practice to a considerable extent by the effect of hydraulic friction and of the slip of water over the surfaces of the impeller and pump casing.

The peripheral velocity of impeller for no delivery of water, in pumps of Cases 3 and 5 has the following value

$$v_2 = \sqrt{\frac{2gh}{2-m^2}},$$

m	·6	·5	·4	·3	·2	·1	·0
$\frac{1}{\sqrt{2-m^2}}$	·781	·756	·738	·724	·714	·709	·707

[TABLE.

Table XVI. *Showing the Efficiency ε for Successive Velocities of Impeller in Forced Vortex Pump, Case 5.*

Discharge angle ϕ_2	Value of x in $\frac{v_2}{=x\sqrt{2gh}}$	Efficiency ε for				
		$m=\cdot6$	$m=\cdot5$	$m=\cdot4$	$m=\cdot3$	$m=\cdot2$
90°	·725				·951	·951
	·75			·889	·889	
	·775		·833	·833		
	·80	·781	·871			
	·85	·692				
75°	·725				·955	·981
	·75			·921	·955	·980
	·775		·877	·917	·950	·976
	·80	·822	·868	·906	·941	·971
	·85	·795	·835			
60°	·75			·923	·957	·981
	·80	·828	·884	·929	·958	·981
	·85	·832	·880	·920	·954	·979
	·90	·821	·867	·909		
45°	·80	·830	·888	·930	·961	·983
	·85	·845	·895	·934	·963	·983
	·90	·851	·896	·933	·962	·983
	·95	·846	·893	·930	·960	·982

Table XVII. *Showing the Coefficient k in the Expression for the Radial Velocity* $u_2 = k\sqrt{2gh}$ *for Successive Velocities of Impeller in Forced Vortex Pump, Case 5.*

Discharge angle ϕ_2	Value of x in $\frac{v_2}{x\sqrt{2gh}}$	Coefficient k for				
		$m = \cdot6$	$m = \cdot5$	$m = \cdot4$	$m = \cdot3$	$m = \cdot2$
90°	·725				·209	·866
	·75			·468	·907	
	·775		·452	·840		
	·80	·371	·693			
	·85	·717				
75°	·725				·012	·081
	·75			·099	·193	·259
	·775		·146	·266	·357	·426
	·80	·150	·299	·412	·507	·583
	·85	·411	·543			
60°	·75			·047	·093	·122
	·80	·079	·161	·221	·255	·282
	·85	·242	·314	·365	·404	·431
	·90	·386	·449	·501		
45°	·80	·047	·096	·128	·150	·164
	·85	·154	·193	·220	·239	·252
	·90	·247	·280	·305	·323	·335
	·95	·328	·361	·384	·402	·414

Centrifugal Pumps having Divergent Vortex Chamber without Guide Vanes. An ordinary free vortex pump can be improved to some extent by making the vortex chamber diverge outwards. But it is evident that a high degree of divergence will not be of much practical value, because reversed currents and eddies may set in along the diverging side walls forming the boundary of vortex motion. By how much a free vortex pump is improved by having a divergent vortex chamber can be seen from the following theory.

The absolute outflow velocity of water from the vortex chamber is given by equation (13), namely,

$$w_3=\frac{r_2}{r_3}\sqrt{\left(\frac{b_2}{b_3}\right)^2 u_2^2+s_2^2}, \quad \ldots\ldots\ldots\ldots\ldots \quad (13)$$

in which s_2 is the tangential component of the absolute outflow velocity from the impeller and is equal to

$$\begin{aligned} s_2 &= w_2 \cos a_2 \\ &= v_2 - c_2 \cos \phi_2, \end{aligned}$$

and u_2, the radial velocity of outflow from the impeller, is

$$\begin{aligned} u_2 &= w_2 \sin a_2 \\ &= c_2 \sin \phi_2. \end{aligned}$$

As in Case 3 it is supposed that no special attention is paid to the form and proportion of the spiral casing, so that the whole of the outflow velocity w_3 is very nearly wasted and we have

$$\begin{aligned} F &= \frac{w_3^2}{2g} \\ &= \frac{m^2}{2g}\left\{n^2(c_2 \sin \phi_2)^2+(v_2-c_2 \cos \phi_2)^2\right\}, \quad \ldots \quad (94) \end{aligned}$$

in which $m=\frac{r_2}{r_3}$ and $n=\frac{b_2}{b_3}$.

Substituting this value of F in (35) the efficiency of the pump becomes

$$\varepsilon = 1 - \frac{m^2\{n^2(z\sin\phi_2)^2 + (1 - z\cos\phi_2)^2\}}{2(1 - z\cos\phi_2)}, \quad \ldots\ldots\ldots \quad (95)$$

in which $$z = \frac{c_2}{v_2}.$$

Differentiating this and equating the result to zero, we obtain a relation between v_2 and c_2 for maximum efficiency,

$$1 - z\cos\phi_2 = \sin\phi_2', \quad \ldots\ldots\ldots\ldots\ldots \quad (96)$$

in which

$$\sin\phi_2' = \frac{n\sin\phi_2}{\sqrt{n^2\sin^2\phi_2 + \cos^2\phi_2}}, \quad \ldots\ldots\ldots\ldots \quad (97)$$

which may be written

$$\tan\phi_2' = n\tan\phi_2. \quad \ldots\ldots\ldots\ldots\ldots \quad (97')$$

The value of the maximum efficiency then becomes

$$\varepsilon = 1 - \frac{m^2\sin\phi_2'}{1 + \sin\phi_2'}. \quad \ldots\ldots\ldots\ldots\ldots \quad (98)$$

Now since

$$\varepsilon = \frac{2gh}{2\,v_2(v_2 - c_2\cos\phi_2)}$$

$$= \frac{2gh}{2\,v_2\,v_2\sin\phi_2'},$$

we have

$$v_2 = \sqrt{\frac{2gh}{2\,\varepsilon\sin\phi_2'}}. \quad \ldots\ldots\ldots\ldots\ldots \quad (99)$$

Other velocities corresponding to the maximum efficiency are found to be

$$c_2 = \frac{1 - \sin\phi_2'}{\cos\phi_2}v_2, \quad \ldots\ldots\ldots\ldots\ldots \quad (100)$$

$$u_2 = w_2 \sin \alpha_2 = c_2 \sin \phi_2$$

$$= \tan \phi_2 (1 - \sin \phi_2') v_2, \quad \ldots\ldots\ldots\ldots\ldots \quad (101)$$

$$w_2 \cos \alpha_2 = v_2 - c_2 \cos \phi_2$$

$$= v_2 \sin \phi_2'. \quad \ldots\ldots\ldots\ldots\ldots \quad (102)$$

The direction of the absolute velocity of inflow into the vortex chamber is given by

$$\tan \alpha_2 = \frac{w_2 \sin \alpha_2}{w_2 \cos \alpha_2} = \frac{u_2}{w_2 \cos \alpha_2}$$

$$= \frac{\tan \phi_2 (1 - \sin \phi_2')}{\sin \phi_2'}. \quad \ldots\ldots\ldots\ldots \quad (103)$$

Now the equation (13) can be written in the form

$$w_3 = \frac{r_2}{r_3} w_2 \sqrt{n^2 \sin^2 \alpha_2 + \cos^2 \alpha_2}.$$

But by (20) we have

$$r_3 b_3 w_3 \sin \alpha_3 = r_2 b_2 w_2 \sin \alpha_2.$$

Hence

$$\sin \alpha_3 = \frac{n \sin \alpha_2}{\sqrt{n^2 \sin^2 \alpha_2 + \cos^2 \alpha_2}}, \quad \ldots\ldots\ldots\ldots \quad (104)$$

which can be written in the form

$$\tan \alpha_3 = n \tan \alpha_2. \quad \ldots\ldots\ldots\ldots\ldots \quad (104')$$

This angle α_3 gives the direction of the outflow velocity of water from the vortex chamber. Since α_3 is smaller than α_2 a particle of water in passing through the vortex chamber from the inner circumference to the outer takes a course which gradually becomes more nearly tangential to the circumference. From equations (98) and (99) it follows at once that so far as the maximum efficiency and the corresponding velocity of the impeller are con-

cerned, the effect of making the vortex chamber diverge outwards is exactly equivalent to reducing the discharge angle from ϕ_2 to ϕ_2' as given by the equation (97′). To take an example. A free vortex pump having a divergent vortex chamber without guide vanes suitable for suction and discharge pipes of 8″ diameter may have the following dimensions:—

Inner dia. of impeller $2r_1$	9″
Outer dia. of impeller $2r_2$	20″
Outer dia. of vortex chamber $2r_3$	27″
Discharge angle of impeller ϕ_2	30°
Axial width of vortex chamber,	
at inflow circumference b_2...	$\frac{7}{8}$″
at outflow circumference b_3...	$1\frac{5}{16}$″

Here
$$\tan\phi_2' = n\tan\phi_2 = \frac{b_2}{b_3}\tan\phi_2$$
$$= \left(\frac{7''}{8} \div 1\frac{5''}{16}\right)\tan 30°$$
$$= \tan 21° 3'.$$

The maximum efficiency is
$$\epsilon = 1 - \frac{m^2 \sin\phi_2'}{1+\sin\phi_2'}$$
$$= 1 - \frac{\left(\frac{20}{27}\right)^2 \sin 21° 3'}{1+\sin 21° 3'} = \cdot 855.$$

The velocity of the impeller for the maximum efficiency is found by (99) to be
$$v_2 = 1\cdot 276\sqrt{2gh}.$$

If in this pump the vortex chamber did not diverge, the maximum

efficiency would be ·817 and the corresponding velocity of the impeller $1{\cdot}106\sqrt{2gh}$. Thus though there is a gain in efficiency of 3·8 per cent., the velocity of the impeller has to be increased by 15½ per cent. It may be noticed that in a vortex chamber without guide vanes, whether divergent or non-divergent, the effect of hydraulic friction in the chamber is to cause a particle of water to take a somewhat shorter and more radial course, tending to diminish the loss due to friction. Guide vanes in the form of equiangular spirals arranged in a non-divergent chamber are worse than without them, first because they are more curved than the real paths which water particles naturally take in moving in a space where there exists friction, and secondly because they introduce more friction surface.

General Equations for Centrifugal Pumps, Friction being taken into Account. In pumps belonging to any of the five cases previously discussed, the loss of head due to hydraulic friction may be taken into account and included in the expression for F. The total loss of head F then consists of five parts as follows :—

1° The head due to the radial velocity of outflow from the vortex chamber into the spiral casing. This is

$$\frac{(m\, c_2 \sin \phi_2)^2}{2g},$$

in which m is either the ratio of r_2 to r_3, or the ratio given by equation (84). If there is no vortex chamber, m is unity and the said head is then the head due to radial velocity of outflow from the impeller into the spiral casing.

2° A part or the whole of the head due to the tangential velocity of outflow from the vortex chamber into the spiral casing. This is

$$i\frac{1}{2g}\left\{m(v_2-c_2\cos\phi_2)\right\}^2.$$

In Cases 1 and 3, i is unity; in Case 5, i may be taken equal to unity; in Cases 2 and 4, i is $\frac{1}{2}$. In Cases 1 and 2, m is unity.

3° The loss of head due to frictional and other hydraulic resistances of water in the vortex chamber. This can be expressed in terms of the head due to the absolute velocity of inflow into the vortex chamber; thus

$$f_1\frac{w_2^2}{2g}.$$

4° The loss of head due to frictional and other hydraulic resistances of water moving over the inner surfaces of the pump casing next the vanes of the impeller. For a pump having an uncased impeller, this loss of head can be expressed by

$$f_1'\frac{w_2^2}{2g}.$$

In a pump having a cased impeller, there exists a certain amount of dead water between the fixed casing of the pump and the impeller casing. This loss of work is of more mechanical nature than hydraulic and the amount of the loss depends among other things on the linear velocity of the impeller. In what follows the impeller is supposed to be uncased.

5° The loss of head due to frictional and other hydraulic resistances of water moving over the surface of the impeller. This is expressed by

$$f_2\frac{c_2^2}{2g}.$$

In a cased impeller having a small number of vanes, the coefficient f_2 will be approximately double that of an uncased impeller of the same size and proportion.

The total loss of head is then the sum of these five losses and is equal to

$$\begin{aligned} F&=\frac{1}{2g}\left\{m^2(c_2\sin\phi_2)^2+i\,m^2(v_2-c_2\cos\phi_2)^2\right\}\\ &\quad+f_1\frac{w_2^2}{2g}+f_1'\frac{w_2^2}{2g}+f_2\frac{c_2^2}{2g}\\ &=\frac{1}{2g}\left\{m^2(\sin^2\phi_2+i\cos^2\phi_2)+f_1+f_1'+f_2\right\}c_2^2\\ &\quad+\frac{1}{2g}(i\,m^2+f_1+f_1')(v_2^2-2\,v_2\,c_2\cos\phi_2)\\ &=\frac{1}{2g}\left\{A\,c_2^2+a(v_2^2-2\,v_2\,c_2\cos\phi_2)\right\}, \qquad \text{..... (105)} \end{aligned}$$

in which

$$A=m^2(\sin^2\phi_2+i\cos^2\phi_2)+f_1+f_1'+f_2, \qquad \text{..... (106)}$$

and

$$a=i\,m^2+f_1+f_1'. \qquad \text{..... (107)}$$

The above equation (105) is obtained by eliminating w_2 by means of (28). The efficiency of the pump is, by (35),

$$\begin{aligned} \epsilon&=1-\frac{2gF}{2\,v_2(v_2-c_2\cos\phi_2)}\\ &=1-\frac{A\,c_2^2+a(v_2^2-2\,v_2\,c_2\cos\phi_2)}{2\,v_2(v_2-c_2\cos\phi_2)}\\ &=1-\frac{A\,z^2+a(1-2\,z\cos\phi_2)}{2(1-z\cos\phi_2)}, \qquad \text{..... (108)} \end{aligned}$$

where $z = c_2 \div v_2$. The condition of maximum efficiency gives the following relation between c_2 and v_2 :—

$$1 - z \cos \phi_2 = \sin \phi_2', \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (109)$$

in which

$$\cos \phi_2' = \cos \phi_2 \sqrt{\frac{a}{A}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (110)$$

The maximum efficiency is then

$$\varepsilon = 1 - \frac{a \sin \phi_2'}{1 + \sin \phi_2'}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (111)$$

and the corresponding best velocity of the circumference of the impeller is

$$v_2 = \sqrt{\frac{2gh}{2 \varepsilon \sin \phi_2'}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (112)$$

Other velocities corresponding to the maximum efficiency are exactly of the same form as those given by (100), (101), and (102), in which the angle ϕ_2 has now the value given by (110). The equation (103) for the direction of the absolute velocity of inflow into the vortex chamber also remains of the same form.

It has been shown by Tables XVI and XVII that when the ratio m is small and the discharge angle ϕ_2 is large, the velocity of impeller may deviate to a certain extent from the best velocity without sensible loss of efficiency. Also in practice the velocity of maximum efficiency may not be convenient on account of too high or too low a value of the radial velocity of water, which the equation (101) gives. For these reasons it is convenient to have two other sets of formulæ giving the efficiency and other quantities, (I) one set corresponding to a given speed of the impeller and (II) the other corresponding to a given radial velocity of water.

(I) Given v_2, to find ε and other Quantities. Suppose that a pump is driven at a given volocity expressed by

$$v_2 = x\sqrt{2gh}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (92)$$

Substituting the general value of F expressed by (105) in the equation (34), we have

$$2gh + A\,c_2^2 + a(v_2^2 - 2\,v_2\,c_2\cos\phi_2) = 2\,v_2(v_2 - c_2\cos\phi_2), \quad \ldots\ldots\ldots \quad (113)$$

which by $u_2 = c_2\sin\phi_2$ and equation (110) can be transformed into

$$u_2^2 + 2\,u_2\,v_2\frac{1-a}{A}\sin\phi_2\cos\phi_2 + v_2^2\frac{(1-a)^2}{A^2}\sin^2\phi_2\cos^2\phi_2 = \frac{\sin^2\phi_2}{A}\left[\frac{1}{a}\left\{1-(1-a)^2\sin^2\phi_2'\right\}v_2^2 - 2gh\right].$$

Solving this equation for the radial velocity u_2, we obtain

$$u_2 = \frac{\sin\phi_2}{\sqrt{a\,A}}\left[\sqrt{\left\{1-(1-a)^2\sin^2\phi_2'\right\}x^2 - a} - (1-a)x\cos\phi_2'\right]\sqrt{2gh}. \quad \ldots\ldots \quad (114)$$

Writing this equation in the form

$$u_2 = k\sqrt{2gh}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (114')$$

the efficiency of the pump becomes, by (35),

$$\varepsilon = \frac{1}{2\,x(x - k\cot\phi_2)}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (115)$$

Equations for other quantities are given in the next set of formulæ **(II)**.

(II) Given u_2, to find ε and other Quantities. From equation (113) eliminate c_2 by means of $u_2 = c_2\sin\phi_2$ and then write u_2 in

the form given by (114′). Solve the resulting equation for v_2 and we have

$$v_2=\left[\sqrt{\frac{1+A\,k^2\,\mathrm{cosec}^2\,\phi_2}{2-a}+\left\{\frac{(1-a)k\cot\phi_2}{2-a}\right\}^2}\right.$$
$$\left.+\frac{(1-a)k\cot\phi_2}{2-a}\right]\sqrt{2gh}, \quad \ldots\ldots \quad (116)$$

which is of the form of the equation (92). The expression for the efficiency is just the same as (115). Other quantities are

$$u_2=c_2\sin\phi_2=w_2\sin\alpha_2, \quad \ldots\ldots \quad (117)$$

$$w_2\cos\alpha_2=v_2-c_2\cos\phi_2$$
$$=v_2-u_2\cot\phi_2, \quad \ldots\ldots \quad (118)$$

$$\tan\alpha_2=\frac{w_2\sin\alpha_2}{w_2\cos\alpha_2}=\frac{u_2}{v_2-u_2\cot\phi_2}. \quad \ldots\ldots \quad (119)$$

Neglecting the thickness of guide vanes, we have

$$m=\frac{b_2\,r_2\sin\alpha_2}{b_3\,r_3\sin\alpha_3}, \quad \ldots\ldots \quad (84)$$

and $$w_3=m\,w_2, \quad \ldots\ldots \quad (58)$$

or $$b_2\,r_2\,w_2\sin\alpha_2=b_3\,r_3\,w_3\sin\alpha_3. \quad \ldots\ldots \quad (20)$$

In a forced vortex pump the tangential component of outflow velocity from the vortex chamber, $w_3\cos\alpha_3$, can easily be made low enough so that it may be equal to the velocity of water in the spiral casing. This condition enables us to fix one quantity in the expression for m. As an approximate value of α_2 to be used in the expression for m, we may use

$$\textit{approximate}\ \ \alpha_2=\frac{1}{2}\left(\frac{\pi}{2}-\phi_2\right). \quad \ldots\ldots \quad (68)$$

More exact value of α_2 should afterwards be calculated by (119).

Taking the thickness of impeller vanes and of guide vanes into account, we have

$$\left.\begin{aligned} Q &= b_1(2\pi r_1 - n_1 t_1 \operatorname{cosec} \phi_1)u_1 \\ &= b_2(2\pi r_2 - n_2 t_2 \operatorname{cosec} \phi_2)u_2 \\ &= b_3(2\pi r_3 - n_3 t_3 \operatorname{cosec} a_3)w_3 \sin a_3, \end{aligned}\right\} \quad \ldots\ldots \quad (120)$$

in which t_1 and t_2 are the thicknesses of impeller vanes and t_3 the thickness of guide vanes, at radii r_1, r_2, and r_3 respectively; and n_1, n_2, and n_3 are their respective numbers. Since guide vanes are conveniently made thicker and more numerous than impeller vanes, another slight correction should be made on the value of the angle a_2. The direction of the relative inflow velocity of water into the impeller at the inner periphery is given by

$$\tan \phi_1 = \frac{u_1}{v_1} = \frac{r_2}{r_1}\frac{u_1}{v_2}. \quad \ldots\ldots\ldots\ldots\ldots \quad (121)$$

The radial velocity of water through the impeller is advantageously made constant, in which case u_1 is of course equal to u_2. The angle ϕ_1 thus calculated should be slightly increased in order to allow for the frictional drag of the influent water round the boss of the impeller in the direction of rotation. It is better to have a small amount of impulsive action on the influent water by the acting faces of the impeller vanes rather than have a tendency to the formation of a vacuous space on the back faces of the impeller vanes.

Friction in Non-divergent Vortex Chamber. The loss of head due to hydraulic friction in a non-divergent vortex chamber may be estimated in the following manner. Suppose that the vortex chamber is an annular space extending from radius r_2 to radius r_3 and of constant axial width b_1, Fig. 7. The motion being

free vortex, any particle of water in the vortex chamber traces out an equiangular spiral APB which makes a constant angle α with the perpendicular to the radius vector. An element of an arc of any plane curve may be expressed thus $ds = \operatorname{cosec} \alpha \, dr$. Consider a tube of stream of water along the equiangular spiral, of a rectangular cross section, the sides of which are a and b, a

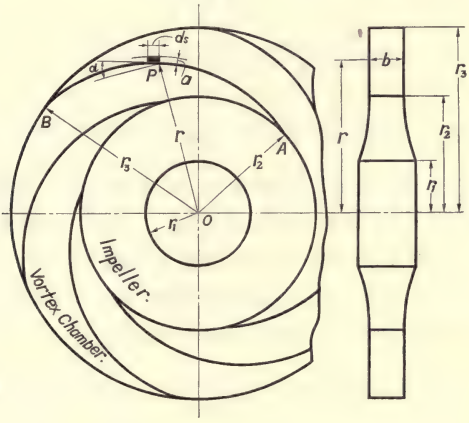


FIG. 7.

being a very small variable width of the tube measured parallel to the bounding planes of the annular space and b the finite thickness of the tube measured parallel to the axis of the pump. The area of a cross of the tube is ab and the friction perimeter is $2a$ and therefore the hydraulic mean depth is $\frac{1}{2}b$. Now the velocity of a water particle along an equiangular spiral in the vortex chamber, at radius r, is

$$w = \frac{r_2}{r} w_2.$$

The loss of head due to friction is

$$\int_{r_2}^{r_3} \mu_1 \frac{ds}{\frac{1}{2}b} \frac{w^2}{2g}$$

$$= \int_{r_2}^{r_3} \mu_1 \frac{\operatorname{cosec} a}{\frac{1}{2}b} \frac{1}{2g} \left(\frac{r_2}{r} w\right)^2 dr$$

$$= 2\,\mu_1 \frac{r_2^2 \operatorname{cosec} a}{b} \frac{w_2^2}{2g} \int_{r_2}^{r_3} \frac{dr}{r^2}$$

$$= 2\,\mu_1 \frac{r_2^2 \operatorname{cosec} a}{b} \frac{w_2^2}{2g} \left(-\frac{1}{r_3} + \frac{1}{r_2}\right)$$

$$= 2\,\mu_1 \frac{r_2}{r_3} \cdot \frac{r_3 - r_2}{b \sin a} \cdot \frac{w_2^2}{2g}, \quad \ldots\ldots\ldots\ldots \quad (122)$$

in which μ_1 is the usual coefficient of hydraulic friction for the wetted surfaces of pipes, conduits, &c. The coefficient f_1, previously given, of hydraulic resistances acting against the entrance and passage of water into and through the vortex chamber can now be estimated by the following formula :—

$$f_1 = \mu_0 + 2\,\mu_1 \frac{r_2}{r_3} \cdot \frac{r_3 - r_2}{b \sin a}, \quad \ldots\ldots\ldots\ldots \quad (123)$$

in which μ_0 is the coefficient of resistance at the entrance of the vortex chamber.

Friction in Divergent Vortex Chamber. The loss of head due to hydraulic friction in a divergent vortex chamber *without guide vanes* may be treated in substantially the same manner as in the case above considered. When *there is a number of guide vanes* in the vortex chamber, the expression for the hydraulic mean depth of a tube of stream in the vortex is a little less than $\frac{1}{2}b$, being

$$\tfrac{1}{2}b\left(1 + \frac{b}{l}\right)^{-1}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (124)$$

in which l is the shortest distance between two consecutive guide vanes. The quantity l varies with the radius r and would be very much greater than b in actual pumps. The bracketed factor may therefore be neglected with but a small amount of error, for which an allowance may be made afterwards by increasing the coefficient of hydraulic friction μ_1 in the proportion of 1 to $\left(1+\frac{b}{l}\right)$, using a mean value of l. This correction of the value of μ_1 being understood, the following formulæ for hydraulic friction in divergent vortex chamber are applicable to pumps with guide vanes as well as to those without guide vanes.

Neglecting the thickness of guide vanes, let the whole areas of the cross sections of the entire current of water forming the vortex motion, at radii r_2, r, and r_3, be denoted by

$$\left.\begin{aligned} A_2 &= 2\pi r_2 b_2 \sin \alpha_2, \\ A &= 2\pi r\ b\ \sin \alpha, \\ A_3 &= 2\pi r_3 b_3 \sin \alpha_3. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (125)$$

Now it is a good plan so to design a vortex chamber that the area A increases uniformly with the radius r from r_2 to r_3 and it is assumed that this is done at least approximately. The variable area A can then be written in the form

$$A = A_0 + B r, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (126)$$

in which $$A_0 = \frac{r_3 A_2 - r_2 A_3}{r_3 - r_2},$$

and $$B = \frac{A_3 - A_2}{r_3 - r_2}.$$

By equation (20) the resultant velocity w of a particle of water in the vortex chamber at any radius r is

$$w = \frac{A_2}{A} w_2.$$

As in the case of a non-divergent vortex chamber, the loss of head due to hydraulic friction is expressed by the following integral

$$\int_{r_2}^{r_3} \mu_1 \frac{ds}{\frac{1}{2}b} \frac{w^2}{2g}$$

$$= 2\,\mu_1 \int_{r_2}^{r_3} \frac{dr}{b \sin \alpha} \left(\frac{A_2}{A}\right)^2 \frac{w_2^2}{2g}$$

$$= 2\,\mu_1\, 2\,\pi\, A_2^2 \frac{w_2^2}{2g} \int_{r_2}^{r_3} \frac{r\,dr}{(A_0 + B\,r)^3}$$

$$= 2\,\mu_1\, 2\,\pi\, A_2^2 \frac{w_2^2}{2g} \frac{(A_3 - A_2)\{2\,A_3\,A_2 - A_2(A_3 + A_2)\}}{2\,A_2^2\,A_3^2\,B^2}$$

$$= \mu_1 \frac{r_2}{r_3} \frac{r_3 - r_2}{b_3 \sin \alpha_3} \left(1 + \frac{b_2 \sin \alpha_2}{b_3 \sin \alpha_3}\right) \cdot \frac{w_2^2}{2g} \cdot \quad \ldots\ldots\ldots\ldots \quad (127)$$

The coefficient f_1, previously given, of hydraulic resistances in a divergent vortex chamber with or without guide vanes therefore becomes

$$f_1 = \mu_0 + \mu_1 \frac{r_2}{r_3} \frac{r_3 - r_2}{b_3 \sin \alpha_3} \left(1 + \frac{b_2 \sin \alpha_2}{b_3 \sin \alpha_3}\right), \quad \ldots\ldots\ldots\ldots \quad (128)$$

in which μ_0 has the same meaning as in (123). One good effect of providing with guide vanes and making b_3 and α_3 larger than b_2 and α_2 is, besides a theoretically calculable gain in efficiency, a considerable diminution of friction head in the vortex chamber. This can be seen by comparing together two formulæ (122) and (127). Suppose for example that a forced vortex pump has the following dimensions :—

Inner dia. of vortex chamber $2r_2$	14″
Outer „ „ „ $2r_3$	20″
Axial width at inner periphery b_2	1″
Axial „ outer „ b_3	1½″

Guide vane angle at inner periphery a_2... ... 14°29′

„ „ „ outer „ a_3... ... 30°

By formula (127) the loss of head due to friction in the vortex chamber is then

$$3{\cdot}73\,\mu_1\frac{w_2^2}{2g}.$$

If this pump be turned into a free vortex pump of the same size, but without divergence of the vortex chamber and without guide vanes, the loss of head due to friction becomes, by (122),

$$16{\cdot}8\,\mu_1\frac{w_2^2}{2g},$$

which is 4·50 times the value above obtained for the forced vortex pump.

Friction over the Fixed Surfaces of the Pump Casing next the Vanes of the Impeller. The loss of head due to this cause can be estimated in an exactly similar manner to that in the case of a divergent vortex chamber. The resulting equation is

$$\mu_1\frac{r_2}{r_1}\frac{r_2-r_1}{b_1\sin a_1}\left(1+\frac{b_2\sin a_2}{b_1\sin a_1}\right)\frac{w_2^2}{2g}, \quad \ldots\ldots \quad (129)$$

in which the angle a_1 for the absolute direction of inflow at the inner periphery of the impeller is 90° or very nearly so. The loss of head due to frictional and other hydraulic resistances over the fixed surfaces of the pump casing next the vanes of the impeller is

$$\mu_0\frac{w_1^2}{2g}+\text{the expression (129)},$$

i.e.,

$$\mu_0\left(\frac{r_2\,b_2\sin a_2}{r_1\,b_1\sin a_1}\right)^2\frac{w_2^2}{2g}+\text{the expression (129)}.$$

Hence the coefficient f_1' previously given becomes

$$f_1' = \mu_0 \left(\frac{r_2 b_2 \sin a_2}{r_1 b_1 \sin a_1} \right)^2 + \mu_1 \frac{r_2}{r_1} \frac{r_2 - r_1}{b_1 \sin a_1} \left(1 + \frac{b_2 \sin a_2}{b_1 \sin a_1} \right), \quad \ldots \quad (130)$$

in which μ_0 is a combined coefficient of resistance of curvature, entrance, change of sectional area, &c. at and near the inner circumference of the pump casing.

Frictional and other Hydraulic Resistances offered to the Entrance and Passage of Water into and through the Impeller. The losses of head due to these causes may be summed in the following form :—

$$f_2 \frac{c_2^2}{2g} = \frac{1}{2g} \left\{ c_1 \sin (\phi_1 - \phi_1') \right\}^2 + \mu_0 \frac{c_1^2}{2g} + \mu_1 \frac{r_2}{r_1} \frac{r_2 - r_1}{b_1 \sin \phi_1} \left(1 + \frac{b_2 \sin \phi_2}{b_1 \sin \phi_1} \right) \frac{c_2^2}{2g}, \quad \ldots\ldots \quad (131)$$

in which ϕ_1' is the calculated value, by equation (121), of the vane angle at the inner periphery of the impeller and ϕ_1 the actual value of that angle which ought to be a little greater than ϕ_1' as has been discussed before. The first term on the right hand side of the above equation represents the loss of head due to sudden change of direction of the flow at the entrance edge of the impeller vanes; the second and third terms have the same meanings as in previous cases. The above equation can be transformed to

$$f_2 \frac{c_2^2}{2g} = \left[\left\{ \sin^2 (\phi_1 - \phi_1') + \mu_0 \right\} \left(\frac{\sin \phi_2}{\sin \phi_1'} \right)^2 + \mu_1 \frac{r_2}{r_1} \frac{r_2 - r_1}{b_1 \sin \phi_1} \left(1 + \frac{b_2 \sin \phi_2}{b_1 \sin \phi_1} \right) \right] \frac{c_2^2}{2g}. \quad \ldots\ldots \quad (132)$$

By calculating the quantity within the square brackets on the

right hand side of this equation an estimated value of the coefficient f_2 previously given can be obtained.

Frictional and Other Hydraulic Resistances in the Spiral Casing. The loss of head due to friction of water in the spiral casing may be calculated in the following manner. The area of a cross section of the spiral casing surrounding the impeller, or the vortex chamber, if there is one, should increase uniformly with the angular distance of the section from the beginning of the spiral casing, so that the velocity of water through the casing may be constant. Referring to Fig. 8, let

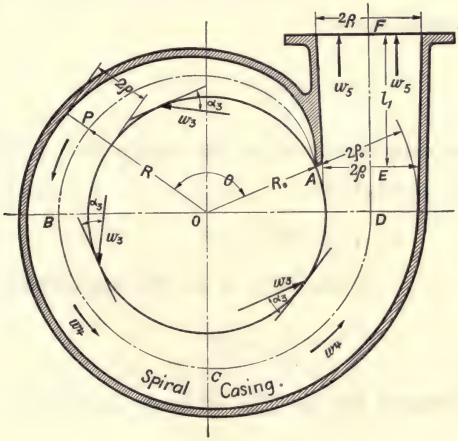


FIG. 8.

ρ=the radius of a cross section of the spiral casing by a radial plane OP at an angular distance,

$\theta = AOP$ measured from the beginning of the casing at A,

R=the radius OP measured from the centre O of the pump to the centre of the section of the spiral casing,

l=the length $APBCDE$ of the centre line of the spiral casing, and let other notations

be as shown in Fig. 8. Then we have

$$\pi\rho^2=\pi\rho_0{}^2\frac{\theta}{2\pi},$$

therefore
$$\rho=\rho_0\left(\frac{\theta}{2\pi}\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (133)$$

and
$$R=R_0+\rho=R_0+\rho_0\left(\frac{\theta}{2\pi}\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots \quad (134)$$

Now the loss of head due to friction of water in the spiral casing can be expressed thus:—

$$\int_0^l 4\mu_1\frac{ds}{2\rho}\frac{w_4^2}{2g},$$

where ds, an elementary arc of the centre line $APBCDE$ of the spiral casing, is equal to

$$ds=\sqrt{(R\,d\theta)^2+(dR)^2}$$

$$=(\text{mean value of } R)\ d\theta, \quad \text{approximately,}$$

$$=\left(R_0+\frac{1}{\sqrt{2}}\rho_0\right)d\theta, \qquad \text{,,}$$

The above integral for the loss of head, therefore, becomes

$$\int_0^{2\pi}4\mu_1\frac{\left(R_0+\frac{1}{\sqrt{2}}\rho_0\right)d\theta}{2\rho_0\left(\frac{\theta}{2\pi}\right)^{\frac{1}{2}}}\frac{w_4^2}{2g}$$

$$=2\times4\mu_1\frac{2\pi\left(R_0+\frac{1}{\sqrt{2}}\rho_0\right)}{2\rho_0}\frac{w_4^2}{2g}$$

$$= 2 \times 4\mu_1 \frac{l}{2\rho_0} \frac{w_4^2}{2g}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (135)$$

But since l is the length of the centre line $APBCDE$ of the spiral casing, it is seen that the friction head in the spiral casing is the same as if the casing were of twice the actual length l and of a constant diameter $2\rho_0$, the velocity w_4 in the casing remaining the same.

The loss of head caused by bends in a pipe, according to a formula given in Hydraulic Machinery by Mr. R.G. Blain, M.E., page 60, may be expressed by

$$(\cdot 0128 + \cdot 0186\, R) \frac{l}{R} \frac{w_4^2}{2g},$$

in which l is the length of the bend or curve measured along the centre line of the pipe, in ft., and R the radius of the bend or curve measured to the centre line in ft. Hence the loss of head due to the combined causes of the friction and curve may be expressed as follows:—

$$f_3 \frac{w_4^2}{2g}$$

$$= \left\{ (\cdot 0128 + \cdot 0186\, R) \frac{l}{R} + 4\mu_1 \frac{l}{\rho_0} \right\} \frac{w_4^2}{2g}, \quad \ldots\ldots \quad (136)$$

in which the mean value of the radius $\left(R_0 + \frac{1}{\sqrt{2}} \rho_0\right)$ may be used for R.

The last portion of a spiral casing is often made into a straight pipe of gradually increasing diameter as shown in Fig. 8. Now loss of head due to hydraulic friction in a straight pipe of conical

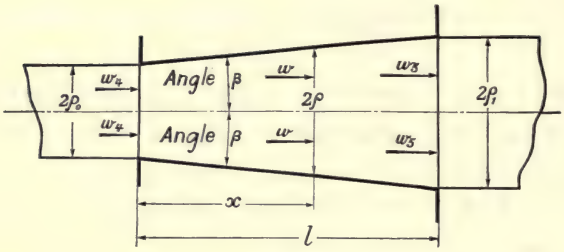


Fig. 9.

shape may be found as follows. Referring to Fig. 9 and using the notations therein indicated, we have

Friction head in conical pipe

$$=\int_0^l 4\mu_1 \frac{\sec\beta\, dx}{2\rho}\frac{w^2}{2g}.$$

But $$\rho^2 w=\rho_0^2 w_4=\rho_1^2 w_5,$$

and $$\rho=\frac{\rho_1-\rho_0}{l}x+\rho_0=x\tan\beta+\rho_0.$$

Substituting these in the above expression and effecting the integration we obtain

Friction head in conical pipe

$$=\tfrac{1}{2}\mu_1 \operatorname{cosec}\beta\left(1-\frac{\rho_0^4}{\rho_1^4}\right)\frac{w_4^2}{2g}$$

$$=\tfrac{1}{2}\mu_1 \operatorname{cosec}\beta\left(\frac{\rho_1^4}{\rho_0^4}-1\right)\frac{w_5^2}{2g}$$

$$=\tfrac{1}{2}\mu_1 \operatorname{cosec}\beta\left(\frac{w_4^2}{2g}-\frac{w_5^2}{2g}\right), \quad \ldots\ldots\ldots\ldots \quad (137)$$

which it is convenient to write in the form

$$\left.\begin{array}{l} f_4\left(\frac{w_4^2}{2g}-\frac{w_5^2}{2g}\right), \\ \text{where} \qquad f_4=\frac{1}{2}\,\mu_1 \operatorname{cosec}\beta. \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (137')$$

Best Velocity in the Spiral Casing. Suppose that the constant velocity w_4 in the spiral casing is lower than the tangential component $w_3 \cos \alpha_3$ of the outflow velocity w_3 from the vortex chamber into spiral casing (or the tangential component $w_2 \cos \alpha_2$ of the outflow velocity w_2 from the impeller into the spiral casing, in the case of a pump without a vortex chamber), then the total loss of head in the spiral casing consists, besides an unavoidable loss due to the radial component $w_3 \sin \alpha_3$, of three parts as follows :—

1.° The loss of head due to sudden change of velocity from $w_3 \cos \alpha_3$ to w_4.

2.° The loss of head due to hydraulic friction in the main portion of the spiral casing, in which the velocity is constant and equal to w_4.

3.° The loss of head due to hydraulic friction in the last portion of the spiral casing, in which the velocity gradually slows down from w_4 to w_5.

The sum of these three losses of head may be expressed in the form :—

$$h_0=\frac{(w_3 \cos \alpha_3-w_4)^2}{2g}+f_3\frac{w_4^2}{2g}+f_4\left(\frac{w_4^2}{2g}-\frac{w_5^2}{2g}\right), \quad \ldots\ldots \quad (138)$$

in which f_3 has the value expressed by equation (136) and f_4 is given by (137'). Supposing that the direction and magnitude of the outflow velocity w_3 from the vortex chamber and also the

velocity w_5 in the delivery pipe are settled beforehand, it is possible so to design the spiral casing that the sum of losses of head may be the least. For this purpose, the velocity w_4 in the spiral casing must satisfy the following condition :—

$$2g\frac{dh_0}{dw_4}=-2(w_3\cos a_3-w_4)+2f_3\,w_4+2f_4\,w_4=0,$$

therefore
$$w_4=\frac{w_3\cos a_3}{1+f_3+f_4}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (139)$$

which gives the best velocity in the spiral casing. The corresponding least value of the sum of losses of head is

$$Least\ h_0=\frac{f_3+f_4}{1+f_3+f_4}\frac{(w_3\cos a_3)^2}{2g}-f_4\frac{w_5^2}{2g}. \quad \ldots\ldots\ldots \quad (140)$$

Now many people believe that by making the velocity of water in the spiral casing equal to the tangential component of the outflow velocity from the vortex chamber (or from the impeller if there be no vortex chamber) and by making the last portion of the spiral casing gradually increase in diameter up to the diameter of the delivery pipe, which is made sufficiently large, there will be the least amount of the loss of head. This is quite otherwise unless $w_3\cos a_3$ be already so low that it is no higher than the velocity w_5 in the delivery pipe. Because if $w_3\cos a_3$ be equal to w_4, the sum of losses in the spiral casing becomes

$$h_0=f_3\frac{(w_3\cos a_3)^2}{2g}+f_4\left\{\frac{(w_3\cos a_3)^2}{2g}-\frac{w_5^2}{2g}\right\}$$

$$=(f_3+f_4)\frac{(w_3\cos a_3)^2}{2g}-f_4\frac{w_5^2}{2g}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (141)$$

Evidently this is always greater than the expression (140). Since f_4 is proportional to $\operatorname{cosec}\beta$ and $\tan\beta$ is proportional to $(\rho_1-\rho_0)$, the f_4 in (141) is necessarily less than the f_4 in (140); but when

compared with f_3, both f_4's are very small and the conclusion above is not much affected. On the other hand since the principal part of the expression (136) for the coefficient f_3 contains ρ_0 as the denominator of a fraction, the f_3 in (141) is necessarily greater than the f_3 in (140) and the conclusion above is emphasized on this account.

Thus a sudden reduction of velocity of water in passing from the vortex chamber (or from the impeller in the case of a pump having no vortex chamber) into the spiral casing means a decidedly less amount of loss of head than when the velocity is made to change gradually in the last portion of the spiral casing and first portion of the delivery pipe. The following example gives an idea of the amount of gain to be obtained by a sudden reduction of velocity. Suppose that we have an 8″ centrifugal pump with an ordinary free vortex chamber of a small size, the principal dimensions of the pump being as follows:—

Dia. of suction and discharge pipes	8″
Outer dia. of impeller	22″
Inner dia. of impeller...	9″
Axial width of impeller at outlet	$\frac{3}{4}$″
Dia. of vortex chamber	24″
Discharge angle ϕ_2 of impeller vanes	30°
Dia. of pump shaft	$2\frac{5}{16}$″
Dia. of space round the boss of impeller	$7\frac{1}{4}$″
Delivery of water per min.	130 cub. ft.
Lift of water by the pump, about	17 ft.

The velocity due to the lift of 17 ft. is equal to $\sqrt{2g \times 17} = 33{\cdot}06$ ft. per second. The velocity of water through the 8″ suction and discharge pipes is 6·217 ft. per second. Let the radial outflow

velocity u_2 from the impeller be equal to the velocity in the suction and discharge pipe; then we have

$$u_2 = \cdot 1880\sqrt{2gh},$$

and $$c_2 = u_2 \operatorname{cosec} \phi_2 = \cdot 3760\sqrt{2gh}.$$

Neglecting, for simplicity, the hydraulic friction through the impeller and vortex chamber and considering the pump to belong to Case 3, we have $i=1$, $f_1=f_1'=f_2=0$, so that (106) and (107) become $A=m^2$ and $a=m^2$, where $m=22\div 24$. Substituting these in the equation (116), the velocity of the outer periphery of the impeller becomes

$$v_2 = 1\cdot 038\sqrt{2gh}.$$

Other velocities are then found as follows:—

$$w_2 \cos \alpha_2 = v_2 - c_2 \cos \phi_2 = \cdot 712\sqrt{2gh},$$

$$\tan \alpha_2 = \frac{u_2}{w_2 \cos \alpha_2} = \cdot 2640 = \tan 14°47',$$

$$w_2 = \cdot 712\sqrt{2gh} \times \sec \alpha_2 = \cdot 7365\sqrt{2gh},$$

$$w_3 = m\, w_2 = \frac{22}{24} w_2 = \cdot 6751\sqrt{2gh},$$

$$\alpha_3 = \alpha_2.$$

Therefore $$w_3 \cos \alpha_3 = \cdot 6527\sqrt{2gh} = 21\cdot 58 \text{ ft. per second.}$$

Now let this tangential component of the outflow velocity from the vortex chamber be suddenly reduced to half the magnitude, so that

$$w_4 = \frac{1}{2} w_3 \cos \alpha_3 = 10\cdot 79 \text{ ft. per second.}$$

Then the diameter of the spiral casing at the end of the main

portion of it in which the velocity is constant and equal to w_4 should be

$$8\sqrt{\frac{6{\cdot}217}{10{\cdot}79}}=6{\cdot}073''=6,''\text{ say.}$$

The radius ρ_0 is therefore 3.″ The mean radius of the centre line of the spiral casing is

$$R=R_0+\frac{1}{\sqrt{2}}\rho_0=12+\frac{1}{\sqrt{2}}\times 3=14{\cdot}12''.$$

Mean circumference $\quad l=2\pi R=88{\cdot}75''.$

The coefficient of hydraulic friction μ_1 for a clean pipe of 6″ diameter according to Darcy is ·0058 and therefore the second term in the expression for f_3 in (136) becomes

$$4\mu_1\frac{l}{\rho_0}=4\times{\cdot}0058\times\frac{88{\cdot}75}{3}={\cdot}686,$$

and the first term in the same expression becomes

$$({\cdot}0128+{\cdot}0186\,R)\frac{l}{R}=\left({\cdot}0128+{\cdot}0186\times\frac{14{\cdot}12}{12}\right)\frac{88{\cdot}75}{14{\cdot}12}={\cdot}109.$$

Hence $\quad f_3={\cdot}109+{\cdot}689={\cdot}795.$

Let the last and conical portion of the spiral casing, in which the diameter gradually increases from 6″ to 8″, be 12″ in length. The coefficient of hydraulic friction for a clean pipe of 7″ mean diameter according to Darcy is ·0057 and therefore

$$f_4=\frac{1}{2}\mu_1\operatorname{cosec}\beta=\frac{1}{2}\times{\cdot}0057\times 12{\cdot}04={\cdot}034.$$

The sum of losses of head due to sudden change of velocity and to hydraulic friction in the spiral casing, as given by (138), becomes

$$h_0=\frac{(w_3\cos a_3-w_4)^2}{2g}+f_2\frac{w_4^2}{2g}+f_4\left(\frac{w_4^2}{2g}-\frac{w_5^2}{2g}\right)$$

$$h_0=(1+f_3+f_4)\frac{w_4^2}{2g}-f_4\frac{w_5^2}{2g}$$

$$=(1+\cdot795+\cdot034)\times\frac{(10\cdot79)^2}{2g}-\cdot034\times\frac{(6\cdot217)^2}{2g}$$

$$=3\cdot310-\cdot020=3\cdot290 \text{ ft.}$$

Besides this the radial velocity of outflow from the vortex chamber is necessarily wasted; its magnitude is

$$w_3 \sin a_3=\cdot6751\sqrt{2gh}\times\sin 14°46'=\cdot1723\sqrt{2gh},$$

and the corresponding loss of head is

$$\frac{(w_3 \sin a_3)^2}{2g}=\cdot504 \text{ ft.}$$

Hence the total loss of head suffered by the water in the act of issuing out from the vortex chamber, going round the spiral casing and discharging into the delivery pipes is $3\cdot29+\cdot50$ $=3\cdot79$ ft.

Now instead of making w_4 equal to half the tangential component of outflow velocity from the vortex chamber, let it be assumed that

$$w_4=w_3 \cos a_3=21\cdot58 \text{ ft. per second.}$$

Calculations exactly similar to the above give the following numbers:—

The radius ρ_0	2·145″
Mean radius R	13·52″
Mean circumference l	84·95″
Coefficient μ_1 for clean pipe 4·29″ dia. ...	·0062
$4\mu_1 l\div\rho_0$	·980
$(\cdot0128+\cdot0186\,R)l\div R$	·211
therefore $f_3=\cdot980+\cdot211=$	1·191,

which is greater by nearly 50 per cent. than the f_3 before obtained. As before let the last and conical portion of the spiral casing be 12″ in length. Then

$$f_4 = \frac{1}{2}\mu_1 \operatorname{cosec}\beta = \frac{1}{2} \times \cdot 0058 \times 6 \cdot 561 = \cdot 019,$$

which is less by 44 per cent. than the f_4 before obtained. Hence the total loss of head suffered by the water in the act of issuing out from the vortex chamber, going round the spiral casing and discharging into the delivery pipe is

$$\frac{(w_3 \sin a_3)^2}{2g} + \text{the expression (141)}$$

$$= \frac{(w_3 \sin a_3)^2}{2g} + (f_3 + f_4)\frac{(w_3 \cos a_3)^2}{2g} - f_4 \frac{w_5^2}{2g}$$

$$= \cdot 504 + (1 \cdot 191 + \cdot 019) \times \frac{(21 \cdot 58)^2}{2g} - \cdot 019 \times \frac{(6 \cdot 217)^2}{2g}$$

$$= \cdot 504 + 8 \cdot 753 = 9 \cdot 257 = 9 \cdot 26 \text{ ft.}$$

Compare this result with the corresponding loss of head before obtained, namely, 3·79 ft.

Motion of a Water Particle in a Vortex Chamber without Guide Vanes, Hydraulic Friction being taken into Account. In a free spiral vortex taking place within a space bounded by two parallel planes, which are perpendicular to the axis of the vortex motion, it is well known that the path of a water particle is an equiangular spiral, hydraulic friction being supposed not to exist. The two bounding surfaces may be curved surfaces so that the space between them diverges gradually and symmetrically about the axis of the motion. A free spiral vortex can take place within such a space and it has been shown before that in this

case the following relation holds, under the same supposition as above :—

$$\tan \alpha = \frac{b_2}{b} \tan \alpha_2, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

and it can be seen easily that

$$\tan \alpha = r \frac{d\theta}{dr}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (14')$$

the notations being as shown in Figs. 2 and 6. Eliminating $\tan \alpha$ between these two equations, we have

$$r \frac{d\theta}{dr} = \frac{b_2}{b} \tan \alpha_2.$$

Hence

$$\theta = b_2 \tan \alpha_2 \int \frac{dr}{br}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (142)$$

If the axial width b be expressed in terms of the radius vector r, the path of a particle of water in the free spiral vortex taking place within the divergent space becomes known by the above equation.

When the resistance due to hydraulic friction in the vortex chamber is taken into account, the path of a water particle becomes shorter and more direct in both of the two cases above considered, discharge angle α_3 becoming greater than when there is no hydraulic friction. This point will now be investigated in detail. Consider a thin cylindrical element of water, of radius r, thickness dr, and axial width b. The centrifugal force of this cylindrical element is

$$\frac{\rho}{g} 2\pi r b \, dr \frac{(w \cos \alpha)^2}{r},$$

in which ρ is the weight of water per cubic unit. This centrifugal force is balanced by three forces :—(a) the increase of

pressure in passing from the inner surface of the thin cylinder to the outer, (*b*) the inertia force of the radial acceleration of water, and (*c*) the hydraulic friction. Now the force (*a*) is

$$2\pi r b\, dp,$$

the force (*b*) is

$$\frac{\rho}{g} 2\pi r b\, dr \frac{d(w \sin a)}{dt},$$

and the force (*c*) is found as follows. The loss of head due to hydraulic friction in passing over an elementary arc ds of the path of a particle of water at any point P is

$$\mu \frac{ds}{R} \frac{w^2}{2g},$$

in which R is the hydraulic mean depth = the cross sectional area of a tube of stream $APB \div$ the friction perimeter = ($b \times$ width of the tube) $\div$ ($2 \times$ width of the tube) = $\frac{1}{2}b$, Fig. 6. In any plane curve $dr = ds \sin a$. Hence the expression above for the loss of head becomes

$$2\mu \frac{dr}{b \sin a} \frac{w^2}{2g}.$$

Multiplying this by $2\pi r b \rho$ and taking the radial component of the result, we obtain the force of friction acting on the elementary cylinder in radial direction,

$$\rho\, 2\pi r b\, 2\mu \frac{dr}{b} \frac{w^2}{2g}.$$

Hence we have

$$\frac{\rho}{g} 2\pi r b\, dr \frac{(w \cos a)^2}{r} = 2\pi r b\, dp + \frac{\rho}{g} 2\pi r b\, dr \frac{d(w \sin a)}{dt} + \rho\, 2\pi r b\, 2\mu \frac{dr}{b} \frac{w^2}{2g},$$

from which we have

$$\frac{d\rho}{dr}=\frac{\rho}{g}\left\{\frac{(w\cos\alpha)^2}{r}-\frac{d(w\sin\alpha)}{dt}-\frac{\mu}{b}w\right\}. \quad \ldots\ldots\ldots\ldots \quad (143)$$

Now Bernoulli's theorem gives

$$\frac{p}{\rho}+\frac{w^2}{2g}+\int_{r_2}^{r}2\mu\frac{dr}{b\sin\alpha}\frac{w^2}{2g}=\text{const.},$$

which by differentiation becomes

$$\frac{1}{\rho}\frac{dp}{dr}+\frac{w}{g}\frac{dw}{dr}+\frac{2\mu}{b\sin\alpha}\frac{w^2}{2g}=0. \quad \ldots\ldots\ldots\ldots\ldots \quad (144)$$

Eliminating $\frac{dp}{dr}$ between this equation and (143), we have

$$\frac{(w\cos\alpha)^2}{r}-\frac{d(w\sin\alpha)}{dt}-\frac{\mu}{b}w^2+w\frac{dw}{dr}+\frac{\mu}{b\sin\alpha}w^2=0.$$

But $$w\frac{dw}{dr}=\frac{1}{2}\frac{d}{dr}(w^2),$$

and $$\frac{d(w\sin\alpha)}{dt}=\frac{d(w\sin\alpha)}{dr}\frac{dr}{dt}$$

$$=w\sin\alpha\frac{d}{dr}(w\sin\alpha)$$

$$=\frac{1}{2}\frac{d}{dr}(w\sin\alpha)^2.$$

The above equation therefore becomes

$$\frac{(w\cos\alpha)^2}{r}+\frac{1}{2}\frac{d}{dr}w^2-\frac{1}{2}\frac{d}{dr}(w\sin\alpha)^2+\frac{\mu}{b}\left(\frac{1}{\sin\alpha}-1\right)w^2=0,$$

i.e., $$\frac{1}{2}\frac{d(w\cos\alpha)^2}{dr}+\frac{(w\cos\alpha)^2}{r}+\frac{\mu}{b}\frac{(w\cos\alpha)^2}{\sin\alpha(1+\sin\alpha)}=0,$$

i.e., $$\frac{d(w\cos\alpha)}{dr}+\frac{w\cos\alpha}{r}+\frac{\mu\, w\cos\alpha}{b\sin\alpha(1+\sin\alpha)}=0,$$

therefore $$\frac{d(w\cos\alpha)}{(w\cos\alpha)}+\frac{dr}{r}+\frac{\mu\, dr}{b\sin\alpha(1+\sin\alpha)}=0.$$

Integrating we obtain

$$\log (w \cos \alpha) + \log r + \int \frac{\mu\, dr}{b \sin \alpha (1 + \sin \alpha)} = \text{const.}$$

Therefore $$r\, w \cos \alpha\, e^{P} = \text{const.} = r_2\, w_2 \cos \alpha_2\, e^{P_2}, \quad \ldots\ldots\ldots \quad (145)$$

in which $$P = \int \frac{\mu\, dr}{b \sin \alpha (1 + \sin \alpha)}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (146)$$

In any vortex whether free or forced, we have

$$b\, r\, w \sin \alpha = \text{const.} = b_2\, r_2\, w_2 \sin \alpha_2. \quad \ldots\ldots\ldots \quad (20)$$

Dividing this by (145), member by member, we obtain

$$b \tan \alpha\, e^{-P} = \text{const.} = b_2 \tan \alpha_2\, e^{-P_2}. \quad \ldots\ldots\ldots \quad (147)$$

Taking the logarithms of the members of this equation, we have

$$\log(b \tan \alpha) - P = \text{const.} = \log(b_2 \tan \alpha_2) - P_2. \quad \ldots \quad (147')$$

If the axial width b be given in terms of the radius vector r, the path of a water particle in a vortex chamber under the influence of hydraulic friction becomes known by (147) or (147'). The direction of motion of a water particle is defined by the angle α, which can be found as follows.

The Angle α for a Vortex Chamber of constant axial Width provided with no Guide Vanes. Suppose that the vortex takes place within a space bounded by two parallel planes. In this case the axial width b is constant and equal to b_2. Differentiating (147') with respect to r, we have

$$\frac{1}{\tan \alpha} \frac{1}{\cos^2 \alpha} \frac{d\alpha}{dr} - \frac{dP}{dr} = 0,$$

i.e., $$\frac{1}{\sin \alpha \cos \alpha} \frac{d\alpha}{dr} - \frac{\mu}{b_2 \sin \alpha (1 + \sin \alpha)} = 0,$$

which gives $$\frac{1+\sin\alpha}{\cos\alpha}d\alpha-\frac{\mu}{b_2}dr=0.$$

Integrating we obtain

$$-2\log\sin\frac{1}{2}\left(\frac{\pi}{2}-\alpha\right)-\frac{\mu}{b_2}r=\text{const.}$$

Hence $$e^{\frac{\mu r}{2b_2}}\sin\frac{1}{2}\left(\frac{\pi}{2}-\alpha\right)=\text{const.}$$

$$=e^{\frac{\mu r_2}{2b_2}}\sin\frac{1}{2}\left(\frac{\pi}{2}-\alpha_2\right). \quad \ldots \quad (148)$$

$$\sin\frac{1}{2}\left(\frac{\pi}{2}-\alpha_3\right)=e^{-\frac{\mu(r_2-r_3)}{2b_2}}\sin\frac{1}{2}\left(\frac{\pi}{2}-\alpha_2\right). \quad \ldots \quad (149)$$

This equation enables us to calculate the discharge angle α_3 of a particle of water issuing out from the vortex chamber.

The Angle α for a Divergent Vortex Chamber, without Guide Vanes. Suppose that the vortex takes place within a space which diverges gradually outwards from the axis of the motion, Fig. 2. In this case an approximate solution can be found in the following manner. Differentiating (147′), we have

$$\frac{1}{b\tan\alpha}\frac{d}{dr}(b\tan\alpha)-\frac{\mu}{b\sin\alpha(1+\sin\alpha)}=0,$$

$$\frac{d}{dr}(b\tan\alpha)-\frac{\mu}{\cos\alpha(1+\sin\alpha)}=0.$$

The value of the denominator $\cos\alpha\,(1+\sin\alpha)$ of the fractional term in this equation is unity when the angle α is zero; it is maximum and equal to 1·30 when α is 30°. In a given vortex chamber of constant axial width b, the variation of the angle α from α_2 to α_3 amounts to only a few degrees and in the case of a divergent vortex chamber, since $\tan\alpha_3$ is a little greater than

$(b_2 \tan a_2) \div b_3$ and a_2 probably never exceeds about 30°, the said variation of a will be small, being somewhat less than

$$\left(1-\frac{b_2}{b_3}\right)a_2.$$

Besides, the value of μ is known approximately only. For these reasons, it is sufficient for our purpose to consider $\cos a(1+\sin a)$ as an approximately constant quantity. Integrating the above equation with this assumption, we have

$$b \tan a - \frac{\mu r}{\cos a\,(1+\sin a)} = \text{const.}$$

$$= b_2 \tan a_2 - \frac{\mu r_2}{\cos a_2(1+\sin a_2)}. \quad \ldots\ldots\ldots \quad (150)$$

To find the value of a_3, a first approximation may be made, thus :—

$$\tan a_3' = \frac{1}{b_3}\left\{b_2 \tan a_2 + \frac{\mu(r_3-r_2)}{\cos a_2(1+\sin a_2)}\right\}. \quad \ldots \quad (151)$$

With this value a second approximation may be made, thus :—

$$\tan a_3'' = \frac{1}{b_3}\left\{b_2 \tan a_2 + \frac{\mu r_3}{\cos a_3'(1+\sin a_3')} - \frac{\mu r_2}{\cos a_2(1+\sin a_2)}\right\}. \quad \ldots \quad (151')$$

Friction in a Vortex Chamber without Guide Vanes. The total loss of head due to hydraulic friction in a divergent or non-divergent vortex chamber without guide vanes is expressed by the following integral :—

$$\int_{r_2}^{r_3} 2\mu \frac{dr}{b \sin a} \frac{w^2}{2g},$$

which by the elimination of w by means of (20) becomes

$$2\mu(b_2 r_2 \sin a_2)^2 \frac{w_2^2}{2g} \int_{r_2}^{r_3} \frac{r\,dr}{(b\,r \sin a)^3}.$$

If the axial width b be given in terms of the radius vector r, the angle a can be expressed as a function of r, at least theoretically, by means of (148) or (150). The integration above indicated may then become possible; it is however accurate enough for our purpose to assume that the quantity $b\,r \sin a$ is a linear function of r as has been done in the equation (126). The result is of course the same as the equation (127), in which the discharge angle a_3 should now have the value given by (149) or (151′).

Numericial Example I. Given a non-divergent vortex chamber without guide vanes in which $r_2=7''$, $r_3=10''$, $b_2=b=b_3=\frac{1}{2}''$ and $a_2=15°$, to find the discharge angle a_3 and the loss of head due to friction in the vortex chamber.

Applying Darcy's formula for the coefficient of hydraulic friction in clean iron pipes, the value of μ for the vortex chamber becomes 0·010. If we neglect the effect of friction in altering the direction of the course of streams in the vortex chamber, the discharge angle a_3 would be the same as a_2, that is, 15°. This value of a_3 gives an approximate value of the friction head in the vortex chamber equal to the expression (122), namely,

$$2\mu \frac{r_2}{r_3} \frac{r_3 - r_2}{b_2 \sin a_2} \frac{w_2^2}{2g}$$

$$= 2\mu \times \frac{7}{10} \times \frac{10-7}{\frac{1}{2} \times \sin 15°} \times \frac{w_2^2}{2g}$$

$$= 32{\cdot}4\mu \frac{w_2^2}{2g} = 0{\cdot}324 \times \frac{w_2^2}{2g}.$$

Taking into account the small alteration of the discharge angle a_3 by the influence of friction, the equation (149) gives

$$\sin\frac{1}{2}\left(\frac{\pi}{2}-a_3\right)=e^{-\frac{\mu(r_3-r_2)}{2b_3}}\sin\frac{1}{2}\left(\frac{\pi}{2}-a_2\right)$$

$$=e^{-\frac{\cdot010(10-7)}{2\times\frac{1}{2}}}\sin\tfrac{1}{2}(90°-15°)$$

$$=0{\cdot}5908=\sin 36°13'.$$

Hence $$a_3=17°34',$$

and the corresponding friction head is

$$\mu\frac{r_2}{r_3}\frac{r_3-r_2}{b_3\sin a_3}\left(1+\frac{b_2\sin a_2}{b_3\sin a_3}\right)\frac{w_2^2}{2g}$$

$$=0{\cdot}010\times\frac{7}{10}\times\frac{10-7}{\frac{1}{2}\sin 17°34'}\times\left(1+\frac{\frac{1}{2}\sin 15°}{\frac{1}{2}\sin 17°34'}\right)\frac{w_2^2}{2g}$$

$$=0{\cdot}265\times\frac{w_2^2}{2g}.$$

This result is 18 per cent. less than the above approximate value and serves to confirm the statement previously made, namely, guide vanes in the form of equiangular spirals arranged in non-divergent vortex chamber are worse than without them.

Numerical Example 2. Given a divergent vortex chamber without guide vanes, in which $r_2=7''$, $r_3=10''$, $b_2=\frac{1}{2}''$, $b_3=1''$ and $a_2=15°$, to find the discharge angle a_3 and the loss of head due to friction in the vortex chamber.

Applying Darcy's formula for the coefficient of friction in clean iron pipes, the value of μ for the vortex chamber becomes 0·0083. Neglecting the small alteration of direction of streams caused by hydraulic friction, the discharge angle a_3 is given by

$$\tan \alpha_3 = \frac{b_2}{b_3} \tan \alpha_2 = \frac{\frac{1}{2}}{1} \times \tan 15° = \tan 7°38',$$

and the corresponding loss of head due to hydraulic friction is

$$\mu \frac{r_2}{r_3} \frac{r_3 - r_2}{b_3 \sin \alpha_3} \left(1 + \frac{b_2 \sin \alpha_2}{b_3 \sin \alpha_3}\right)$$

$$= 0{\cdot}0083 \times \frac{7}{10} \times \frac{10-7}{1 \times \sin 7°38'} \left(1 + \frac{\frac{1}{2} \times \sin 15°}{1 \times \sin 7°38'}\right) \frac{w_2^2}{2g}$$

$$= 31{\cdot}2 \times 0{\cdot}0083 \times \frac{w_2^2}{2g} = 0{\cdot}259 \times \frac{w_2^2}{2g}.$$

Taking into account the small change of direction of streams caused by hydraulic friction, the discharge angle α_3 is given by (151),

$$\tan \alpha_3' = \frac{1}{b_3} \left\{ b_2 \tan \alpha_2 + \frac{\mu (r_3 - r_2)}{\cos \alpha_2 (1 + \sin \alpha_2)} \right\}$$

$$= \frac{1}{1} \left\{ \frac{1}{2} \tan 15° + \frac{{\cdot}0083(10-7)}{\cos 15°(1 + \sin 15°)} \right\}$$

$$= 0{\cdot}1553 = \tan 8°50',$$

and the corresponding loss of head due to friction is

$$\mu \frac{r_2}{r_3} \frac{r_3 - r_2}{b_3 \sin \alpha_3} \left(1 + \frac{b_2 \sin \alpha_2}{b_3 \sin \alpha_3}\right) \frac{w_2^2}{2g}$$

$$= 0{\cdot}0083 \times \frac{7}{10} \times \frac{10-7}{1 \times \sin 8°50'} \times \left(1 + \frac{\frac{1}{2} \times \sin 15°}{1 \times \sin 8°50'}\right) \frac{w_2^2}{2g}$$

$$= 0{\cdot}0083 \times 24{\cdot}8 \frac{w_2^2}{2g} = 0{\cdot}206 \frac{w_2^2}{2g}.$$

So far as frictional resistance is concerned, the loss of head in this divergent vortex chamber is 22 per cent. less than that in a non-divergent vortex chamber of otherwise the same form and dimensions as given in the preceding example.

Centrifugal Pumps having Divergent Vortex Chamber without Guide Vanes, Hydraulic Friction being taken into Account. Assuming that no special attention is paid to the form and proportions of the spiral casing, the whole of the outflow velocity w_3 from vortex chamber is very nearly wasted and the loss of head due to this waste of velocity is given by the equation (94). Taking friction into account, the total loss of head consists of terms quite similar to those in the equation (105); thus

$$F=\frac{w_3^2}{2g}+f_1\frac{w_2^2}{2g}+f_1'\frac{w_2^2}{2g}+f_2\frac{c_2^2}{2g}$$

$$=\frac{m^2}{2g}\left\{n^2(c_2\sin\phi_2)^2+(v_2-c_2\cos\phi_2)^2\right\}$$

$$+(f_1+f_1')\frac{1}{2g}(c_2^2+v_2^2-2\,c_2\,v_2\cos\phi_2)+f_2\frac{c_2^2}{2g}$$

$$=\frac{1}{2g}\left\{A\,c_2^2+a(v_2^2-2\,v_2\,c_2\cos\phi_2)\right\}, \quad \ldots\ldots\ldots \quad (152)$$

in which $$A=m^2(n^2\sin^2\phi_2+\cos^2\phi_2)+f_1+f_1'+f_2, \quad \ldots\ldots\ldots \quad (153)$$

and $$a=m^2+f_1+f_1', \quad \ldots\ldots\ldots\ldots \quad (154)$$

the two quantities m and n being given by

$$m=\frac{r_2}{r_3} \quad \text{and} \quad n=\frac{b_2}{b_3}.$$

Having obtained the three equations above, all the subsequent reasoning and the equations that can be derived therefrom will be the same as those under the heading, "General Equations for Centrifugal Pumps, Friction being taken into Account." All the equations from (108) to (121) are therefore applicable to the present case, the values of A and a being now given by (153) and (154) instead of by (106) and (107).

Impeller Vanes and Guide Vanes made Straight. By probable choice, calculation, and repeated trials of the entrance and discharge angles ϕ_1 and ϕ_2 and of the inner and outer radii r_1 and r_2 of the impeller, it is possible to make the vanes of the impeller straight. Straight vanes, besides being simple in construction, have the advantage of causing the least amount of loss in hydraulic friction. The following geometrical condition must be satisfied for straight vanes of the impeller :—

$$\frac{\cos \phi_1}{\cos \phi_2} = \frac{r_2}{r_1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (155)$$

A similar statement to the above is applicable to the guide vanes and the geometrical condition that must be satisfied for straight guide vanes is as follows :—

$$\frac{\cos \alpha_2}{\cos \alpha_3} = \frac{r_3}{r_2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (156)$$

Tokyo, October, 1904.

XXIII.

Results of Tests of a Forced Vortex Centrifugal Pump.

[*The Journal of the College of Engineering, Tokyo Imperial University, Japan, Vol. II, No. 4, March,* 1905. *The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. VIII, No.* 12, *May,* 1905.]

Tests of 7″ Forced Vortex Centrifugal Pump. The results of tests of a 7″ forced vortex centrifugal pump made at Shibaura Engineering Works, Tokyo, according to the design of the writer, are given in Table XVIII. As shown in Fig. 10, this pump is a non-compound pump having a divergent vortex chamber, which is 1·4 times in outer diameter and 1½ times in outer axial width, of the impeller and in which are arranged seven guide vanes for reducing the velocity of water very quickly and converting it into pressure. Other dimensions and the theoretical maximum efficiency together with the corresponding peripheral velocity of the impeller are given in the foregoing Paper under the heading "**Case 5.**" The pump was designed to deliver 112 cubic feet of water per minute against an actual lift of 120 ft., at an estimated speed of 1150 revolutions per minute. The pump shaft is directly coupled to a three phase alternating current motor of 50 horse power normal, at 220 volts. The numbers under the

Table XVIII. Tests of 7″

Tests made at Shibaura Engineering Works, Tokyo. Impeller, that at inlet 2⅜″; suction and delivery 7″ dia.; lift for

Observation No.	Load on motor per cent.	Volt	Ampere	Motor output H.P.	Gauge head, ins.
					TEST
1	32	220	113	42·5	8½
2	57	,,	110	41·0	7
3	71	,,	105	39·2	6⅛
4	69	,,	98	36·5	5⅜
5	68	,,	90	33·2	5⅛
					TEST
1	69	,,	88	33·3	5 7/16
2	68	,,	85	32·2	5⅛
3	64	,,	95	36·0	6
					TEST
1	150	,,	147·5	57·0	7
2	143	,,	145·0	56·0	6¾
3	102	,,	120·0	46·5	5 3/16

Forced Vortex Centrifugal Pump.

outer dia. $14\frac{1}{4}''$, *inner dia.* $7\frac{1}{8}''$; *axial width at outlet* $\frac{5}{8}''+\frac{1}{64}''$, *which designed* 120 *ft.; delivery* 112 *cub. ft. per minute.*

notch breadth, ins.	Delivery, cub. ft. per min.	Equivalent lift, ft.	Revols. per min.	Effective H.P.	Efficiency per cent.
No. I.					
24	239·1	19·5	1160	8·8	20·7
„	178·6	47·5	1150	16·0	39·0
„	146·1	70·5	1156	19·4	49·6
„	120·5	83·5	1190	19·0	52·0
„	112·0	88·5	1180	18·7	56·4
No. II.					
„	121·3	83·2	1170	19·0	56·7
„	112·0	87·9	1170	18·7	58·0
„	140·7	65·9	1150	17·5	48·3
No. III.					
„	177·3	127·5	1420	42·7	75·0
„	167·9	124·0	1400	39·3	70·0
„	113·1	131·0	1400	28·0	60·0

heading "Motor Output, H.P." in Table XVIII are the effective powers given out by the motor shaft. These numbers were calculated by causing the motor to drive a dynamo of known efficiency and measuring the power given out by the dynamo. The pressure head against which the pump worked was created by throttling the water in the delivery pipe by means of a stop valve and the pressure thus produced was measured by a good standard pressure gauge, which had been carefully tested beforehand. Converting this pressure in lbs. per sq. inch into head of water in ft. and adding the actual height of the centre of the pressure gauge above the level of the suction water, there is obtained the equivalent lift of water by the pump in ft. as given in Table XVIII. The water discharged from the delivery pipe was caused to pass over the sill of a rectangular gauge notch 24 inches in breadth constructed of boards. The velocity of approach of water was quite considerable, but the effect of it was not taken into account in estimating the rate of delivery. The formula used for the calculation was the so-called simple *theoretical* formula,

$$Q = \mu \frac{2}{3} bh \sqrt{2gh},$$

in which the coefficient μ was assumed constant and equal to 0·625. This assumed value of the coefficient is probably slightly too high, giving an over estimate of the rate of delivery. But it is judged that the amount of the under estimate consequent on the neglect of the velocity of approach of water much outweighs the over estimate above stated. In Fig. 11, the lifts of water by the pump and the corresponding efficiencies given in Table XVIII are marked by dots within small circles and a

mean curve is drawn showing a probable value of the efficiency of the pump for any lift, say, from about 30 ft. up to 130 ft. It is remarkable that the efficiency of this forced vortex pump rises steadily as the lift of water becomes greater. The diameter of the impeller being $14\frac{1}{4}''$, its peripheral velocity in ft. per second corresponding to a given number of revolutions n per min. is

$$v_2 = \frac{14\frac{1}{4}\,\pi}{12} \times \frac{n}{60} = 0{\cdot}06218\,n.$$

The velocities v_2 thus calculated for all the values of n in Table XVIII are given in the second column of Table XIX, with the corresponding lifts h in the first column. The velocities due to the lifts h are given in the third column and the values of x in the relation $v_2 = x\sqrt{2gh}$ obtained by dividing the numbers in the second column by those in the third are given in the last column. In Fig. 12, the peripheral velocities of the impeller v_2 given in Table XIX corresponding to various lifts h are plotted and a mean curve is drawn shown showing probable values of the peripheral velocities for lifts up to about 130 ft. In Fig. 13, the values of x given in the last column of Table XIX are set off as ordinates with the corresponding lifts h as abscissæ and a mean curve is drawn showing probable values of the coefficient x.

[TABLE.

Table XIX. *Showing the Observed Values of the Peripheral Velocity v_2 and the Coefficient x in the Relation $v_2 = x\sqrt{2gh}$.*

Lift of water by the pump, h in ft.	Peripheral velocity of impeller, $v_2 = 0{\cdot}06218\ n$ in ft. per sec.	Velocity due to the lift h, $\sqrt{2gh}$ in ft. per sec.	Value of the coefficient x in the relation $v_2 = x\sqrt{2gh}$
19·5	72·13	35·41	2·037
47·5	71·50	55·27	1·294
70·5	71·88	67·33	1·068
83·5	73·99	73·27	1·010
88·5	73·37	75·44	·973
83·2	72·75	73·14	·995
87·9	72·75	75·18	·968
65·9	71·50	65·10	1·098
127·5	88·29	90·54	·975
124·0	87·05	89·29	·975
131·0	87·05	91·78	·949

Peripheral Velocity and Efficiency of 7″ Forced Vortex Centrifugal Pump. From the mean curves given in Figs. 11, 12, and 13, mean values of efficiency, peripheral velocity, and coefficient x were measured for lifts differing by 5 ft. from 30 ft. up to 130 ft. Table XX gives the results of these measurements. It is seen from this table that owing to hydraulic friction as well as to slip of water in the pump, the coefficient x in the expression for the peripheral velocity of the impeller is much higher than its theortical value,

$$v_2 = x\sqrt{2gh}$$

$$= \left[\sqrt{\frac{1+m^2k^2\operatorname{cosec}^2\phi_2}{2-m^2} + \left\{\frac{(1-m^2)k\cot\phi_2}{2-m^2}\right\}^2} + \frac{(1-m^2)k\cot\phi_2}{2-m^2}\right]\sqrt{2gh}. \quad \ldots\ldots \quad (116')$$

This equation is obtained as follows. In the equations (106) and (107) in the foregoing Paper make $i=1$, $f_1=f_1'=f_2=0$, then $A=a=m^2$. The substitution of these in (116) gives the above result.

In an ordinary centrifugal pump provided with a small free vortex chamber and having the discharge angle ϕ_2 equal to about 30°, the value of x is usually 1·2 to 1·3, while in the forced vortex pump now being considered the probable actual values of x for lifts higher than about 85 ft. are all less than unity.

Table XX. *Showing Peripheral Velocity of Impeller v_2, Coefficient x in $v_2 = x\sqrt{2gh}$, and Efficiency of the Pump, measured from the Diagrams, Figs.* 12, 13, *and* 11.

Lift of water h in ft.	Velocity of impeller v_2, from Diagram, Fig. 12.	Value of x in $v_2 = x\sqrt{2gh}$, from Diagram, Fig. 13.	Efficiency of the pump, from Diagram, Fig. 11
30	71·8	1·635	28·2
35	71·7	1·510	31·6
40	71·7	1·414	34·7
45	71·6	1·330	37·6
50	71·4	1·259	40·4
55	71·4	1·200	42·9
60	71·4	1·149	45·3
65	71·6	1·107	47·6
70	71·8	1·070	49·7
75	72·1	1·037	51·7
80	72·5	1·011	53·7
85	73·0	·988	55·5
90	73·8	·970	57·3
95	74·8	·959	59·0
100	76·3	·952	60·6
105	77·9	·949	62·1
110	79·7	·948	63·6
115	81·7	·952	65·0
120	84·0	·956	66·4
125	86·5	·963	67·7
130	88·9	·972	69·0

The coefficient x for any lift can be obtained by dividing the peripheral velocity v_2 given in the second column of Table XX by the velocity due to the lift. The result will be found to be the same as those given in the third column of the table.

Radial Velocity of Outflow of Water in 7″ Forced Vortex Centrifugal Pump. The radial velocity of water from the impeller into the vortex chamber in the 7″ pump is given by

$$u_2 = \frac{Q}{b_2(2\pi r_2 - n_2 t_2 \operatorname{cosec} \phi_2)}$$

$$= \frac{\frac{1}{60}(Q \text{ in cub. ft. per min.})}{\frac{1}{12} \times \frac{3}{4} \times \frac{1}{12}\left\{44{\cdot}768 - 6\left(\frac{1}{8} + \frac{1}{32}\right)1{\cdot}0353\right\}}$$

$$= 0{\cdot}073065\, Q.$$

The observed values of the lift h and the delivery Q given in Table XVIII then enable us to calculate the coefficient k in the expression

$$u_2 = k\sqrt{2gh}.$$

In Table XXI the numbers in the second column representing the delivery are the same as those given in Table XVIII; those in the third and fifth columns representing u_2 and k respectively were calculated as indicated above; and those in the fourth column representing the coefficient x are the same as those given in Table XIX. In Fig. 14, the lifts h and the corresponding values of the coefficient k in the equation $u_2 = k\sqrt{2gh}$ are plotted and a mean curve is drawn showing probable values of k for lift up to about 130 ft. In Fig. 15, x and k are set off as abscissa and ordinate respectively and a mean curve is obtained giving a probable relation between these two coefficients. This curve is

almost a straight line. From the diagram, Fig. 13, the lifts h corresponding to successive values of the coefficient x were measured. These lifts are given in the second column of Table XXII. Now from the diagram, Fig. 14, the values of k corresponding to the various lifts h given in the second column of the table were measured and these are given in the third column. The values of k can also be measured from the diagram, Fig. 15, and these are given in the last column of the table. The two sets of values of k thus obtained are almost exactly the same.

Table XXI. *Showing the Observed Values of Delivery Q, Radial Outflow Velocity of Water u_2, and the Coefficient k in the Relation $u_2 = k\sqrt{2gh}$.*

Lift of water by the pump, h in ft.	Delivery Q in cub. ft. per min.	Radial velocity u_2 in ft. per sec.	Value of x in the relation $v_2 = x\sqrt{2gh}$	Value of k in the relation $v_2 = k\sqrt{2gh}$
19·5	239·1	17·47	2·037	·493
47·5	178·6	13·05	1·294	·236
70·5	146·1	10·68	1·068	·1585
83·5	120·5	8·80	1·010	·1230
88·5	112·0	8·18	·973	·1085
83·2	121·3	8·86	·995	·1211
87·9	112·0	8·18	·968	·1089
65·9	140·7	10·28	1·098	·1579
127·5	177·3	12·95	·975	·1431
124·0	167·9	12·27	·975	·1374
131·0	113·1	8·26	·949	·0900

Table XXII. *Showing the Coefficient k in the Expression for Radial Outflow Velocity of Water $u_2 = k\sqrt{2gh}$, measured from the Diagrams, Figs.* 14 *and* 15.

Coefficient x in $v_2 = x\sqrt{2gh}$	Lift h measured from hx diagram, Fig. 13	Coefficient k measured from hk diagram, Fig. 14	Coefficient k measured from xk diagram, Fig. 15
1·35	43·7	·260	·260
1.30	47·0	·240	·240
1·25	50·7	·220	·221
1·20	55·0	·201	·201
1·15	60·0	·181	·181
1·10	65·8	·162	·162
1·05	72·8	·143	·143
1·02	78·0	·131	·131
1·00	82·0	·123	·123
·98	87.3	·116	·115
·97	90.4	·111	·111
·96	94·5	·107	·107
·95	102·0	·103	·104

The relation between x and k as represented by the diagram of tests, Fig. 15, can be expressed by a simple empirical equation :—

$$k = 0{\cdot}39\,x - 0{\cdot}267. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (A)$$

But from equation (114) it can be seen that the true relation between x and k is expressed by the following equation :—

$$k = C_1(\sqrt{x^2 - C_2} - C_3\,x), \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (B)$$

in which C_1, C_2, C_3 are known constants depending on m, ϕ_2, and i and can be accurately calculated by (106), (107), (110), and (114) for hypothetical frictionless pumps. For pumps in actual work, the three constants depends to a large extent also on experimental coefficients of hydraulic friction and it will be found that very different sets of values of these constants can be proposed, all of which give very nearly the same series of values of k as those measured from the diagram of tests in Fig. 14 or Fig. 15. Thus any of the following three equations gives k fairly well agreeing with mean experimental values found from the diagram of tests :—

$$k = 0{\cdot}675(\sqrt{x^2 - 0{\cdot}36} - 0{\cdot}618\,x), \quad \ldots\ldots\ldots \quad (B_1)$$

$$k = 1{\cdot}81(\sqrt{x^2 - 0{\cdot}15} - 0{\cdot}854\,x), \quad \ldots\ldots\ldots\ldots \quad (B_2)$$

and

$$k = 2{\cdot}308(\sqrt{x^2 - 0{\cdot}12} - 0{\cdot}8848\,x). \quad \ldots\ldots\ldots \quad (B_3)$$

The results of calculations by these equations and by the equation (A) are given in Table XXIII, only for the sake of comparison with the values of k given in Table XXII.

Table XXIII. *Showing the Coefficient k in the expression $u_2 = k\sqrt{2gh}$, calculated by Assumed Equations (A), (B_1), (B_2), and (B_3).*

Value of x in $v_2 = x\sqrt{2gh}$	Value of k in $u_2 = k\sqrt{2gh}$ calculated by equation			
	(A)	(B_1)	(B_2)	(B_3)
1·35	·260	·253	·254	·255
1·30	·240	·236	·237	·237
1·25	·221	·219	·219	·219
1·20	·201	·201	·201	·201
1·15	·181	·183	·182	·182
1·10	·162	·163	·163	·163
1·05	·143	·144	·144	·144
1·02	·131	·131	·131	·131
1·00	·123	·123	·123	·123
·99	·119	·119	·119	·119
·98	·115	·114	·115	·114
·97	·110	·110	·110	·110
·96	·107	·106	·106	·105
·95	·104	·101	·102	·102
·94	·100	·096	·097	·097

Table XXIV. Tests of 7″ Forced Vortex Centrifugal Pump.

Further Tests made at Shibaura Engineering Works, Tokyo, of the 7″ Pump, with the Flange Joint of Pump Casing extra-packed, for seeing the Effect of increased Clearance Spaces next the Impeller.

Observation No.	Load on motor per cent.	Volt	Ampere	Motor out put, H.P.	Gauge notch		Delivery, cub. ft. per min.	Equivalent lift., ft.	Revols, per min.	Effective H.P.	Efficiency, per cent.
					head, ins.	breadth, ins.					
TEST No. IV. Thickness of packing 0·096″.											
1	135	220	143	55·0	$7\frac{1}{8}$	24	182·1	108·0	1410	37·1	67·5
2	114	,,	130	50·0	$6\frac{3}{4}$	,,	167·9	99·0	1340	31·4	63·0
3	103	,,	118	45·5	$6\frac{1}{4}$	,,	149·6	99·0	1320	28·3	62·3
4	91	,,	120	46·5	$5\frac{3}{16}$	,,	113·1	117·0	1410	25·0	53·7
TEST No. V. Thickness of packing 0·045″.											
1	126·5	,,	145	56·0	$6\frac{13}{16}$	,,	170·3	108·0	1380	34·8	62·0
2	118·0	,,	140	54·0	$6\frac{1}{2}$	,,	158·7	108·0	1380	32·4	60·0
3	95·7	,,	120	46·5	$5\frac{3}{16}$	,,	113·1	123·0	1400	26·3	56·5
4	94·5	,,	120	46·5	$5\frac{1}{4}$	,,	115·2	119·5	1420	26·0	56·0
5	90·2	,,	117	45·0	$5\frac{3}{8}$	,,	119·3	110·0	1390	24·8	55·0
6	90·0	,,	130	50·0	$6\frac{1}{8}$	,,	145·1	110·0	1405	24·8	49·5
7	89·5	,,	115	44·0	$5\frac{3}{16}$	,,	113·1	115·0	1390	24·8	56·0

Further Tests of 7″ forced Vortex Centrifugal Pump. In the 7″ forced vortex centrifugal pump made at Shibaura Engineering Works, Tokyo, the pump casing, that is, the castings enclosing the impeller and vortex chamber and forming the boundary walls of the spiral passage round the exterior periphery of the vortex chamber, are divided by a central plane perpendicular to the axis of the pump shaft. These castings are joined together by an ordinary flange joint. Now the impeller of this pump being an uncased impeller, it is expected that by making the amount of clearance space between each inner surface of the pump casing and the impeller very small, frictional resistance would be considerable but the slip of water small, while by increasing the clearance space, an opposite effect would be produced, that is, a reduced frictional resistance and an increased amount of the slip of water. With the object of ascertaining definitely, if possible, this point, further tests of the 7″ pump were made at Shibaura Engineering Works. The arrangement and the method of procedure adopted in these tests were just the same as in the previous tests, except that an extra paper packing of thickness equall to 0·096″ was inserted between the flanges of the pump casing in Test No. IV and a similar packing of thickness equal to 0·045″ was inserted between the flanges in Test No. V, each of the two clearance spaces above mentioned being thus increased by half the thickness of the packing used. The results of these tests are given in Table XXIV. The range of the lifts of water by the pump over which the tests extended being unfortunately very limited, it is not possible to draw any definite conclusion from the results. It is however likely that with an increased clearance space the efficiency would fall rather rapidly as the lift becomes higher, but that there would probably

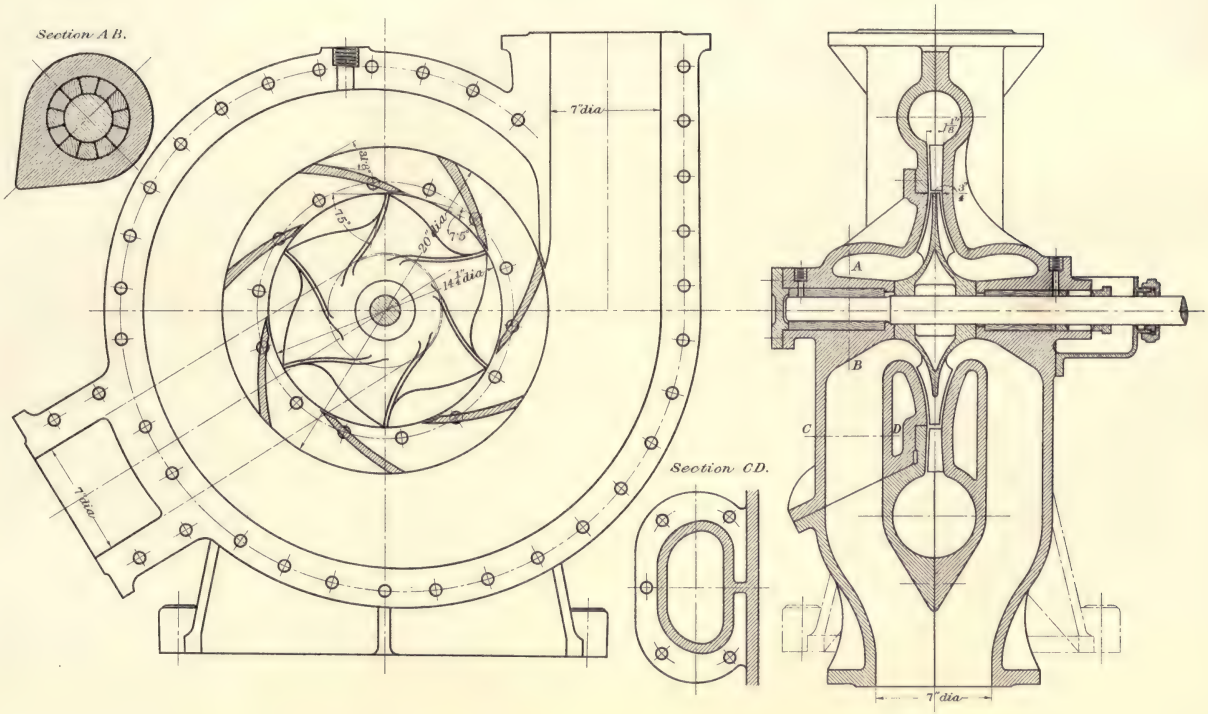
be a slight improvement in efficiency as the lift becomes equal to or lower than about 100 ft.

The writer disires to acknowledge his indebtedness and express his thanks to Mr. Den-ichirō Nishizaki, *Kōgakushi*, mechanical engineer, as well as to other engineers, of Shibaura Engineering Works, Tokyo, for the carrying out of the series of tests, described in this paper, of the 7″ forced vortex centrifugal pump.

Tokyo, December, 1904.

7″ Forced Vortex Centrifugal Pump.

FIG. 10.



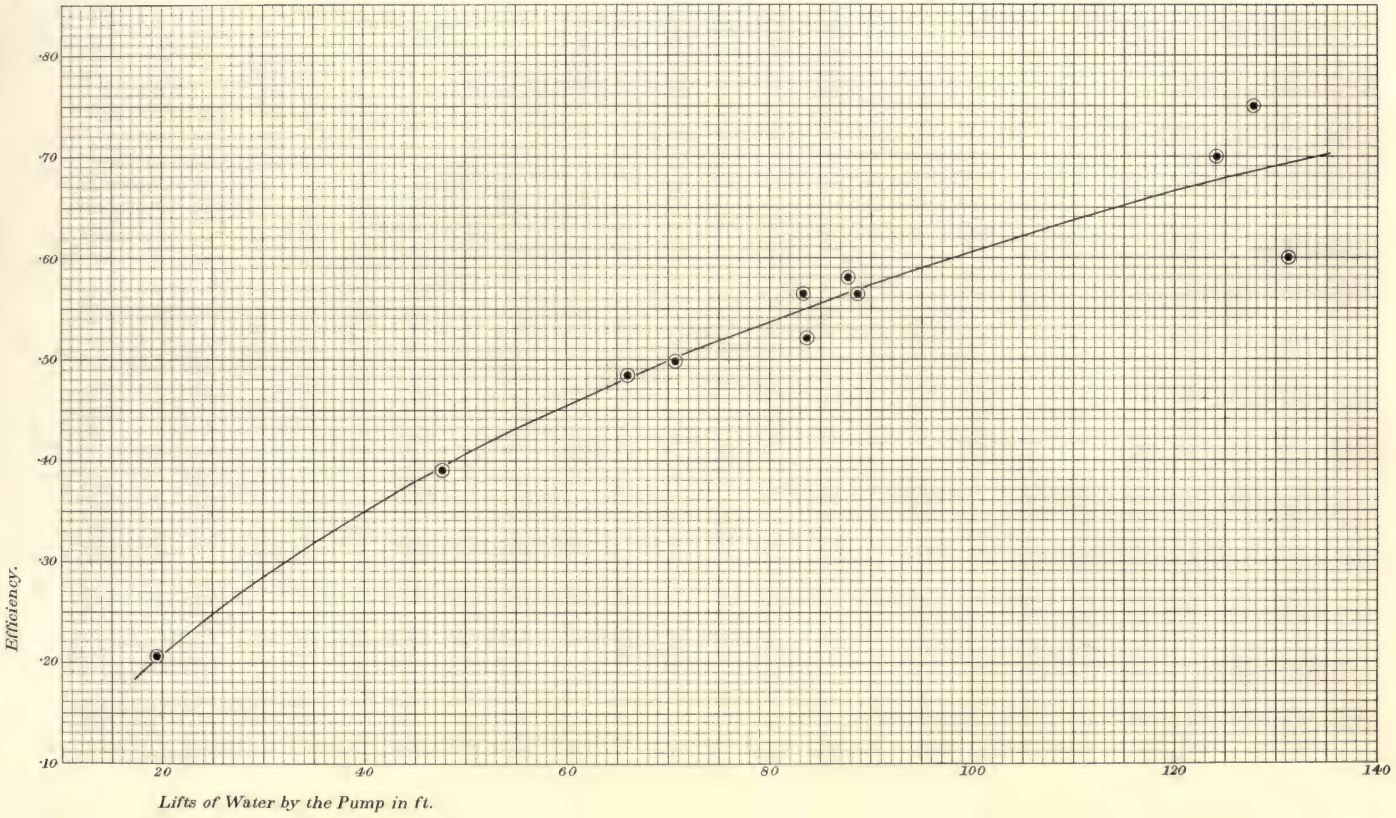
TESTS

OF

7″ FORCED VORTEX CENTRIFUGAL PUMP.

Showing the Efficiency of the Pump for Lifts up to 130 ft.

FIG. 11.



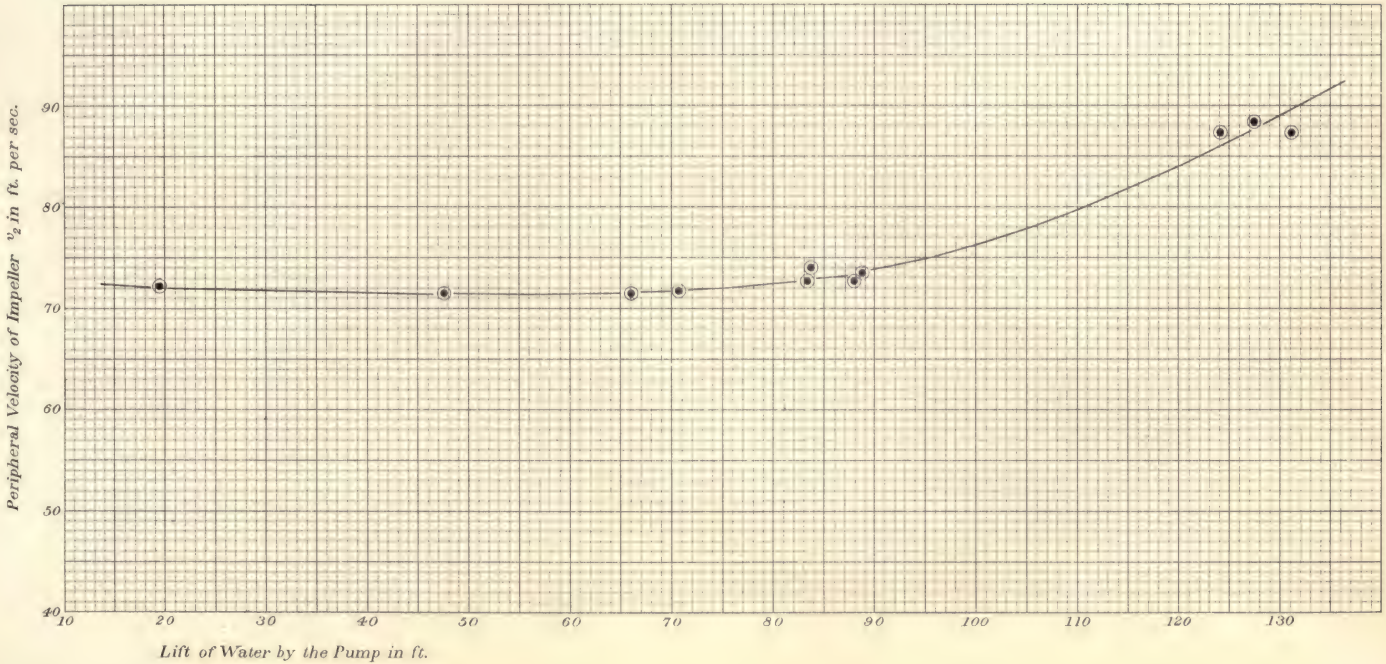
TESTS

OF

7″ FORCED VORTEX CENTRIFUGAL PUMP.

Showing the Peripheral Velocity of the Impeller v_2 for Lift up to 130 ft.

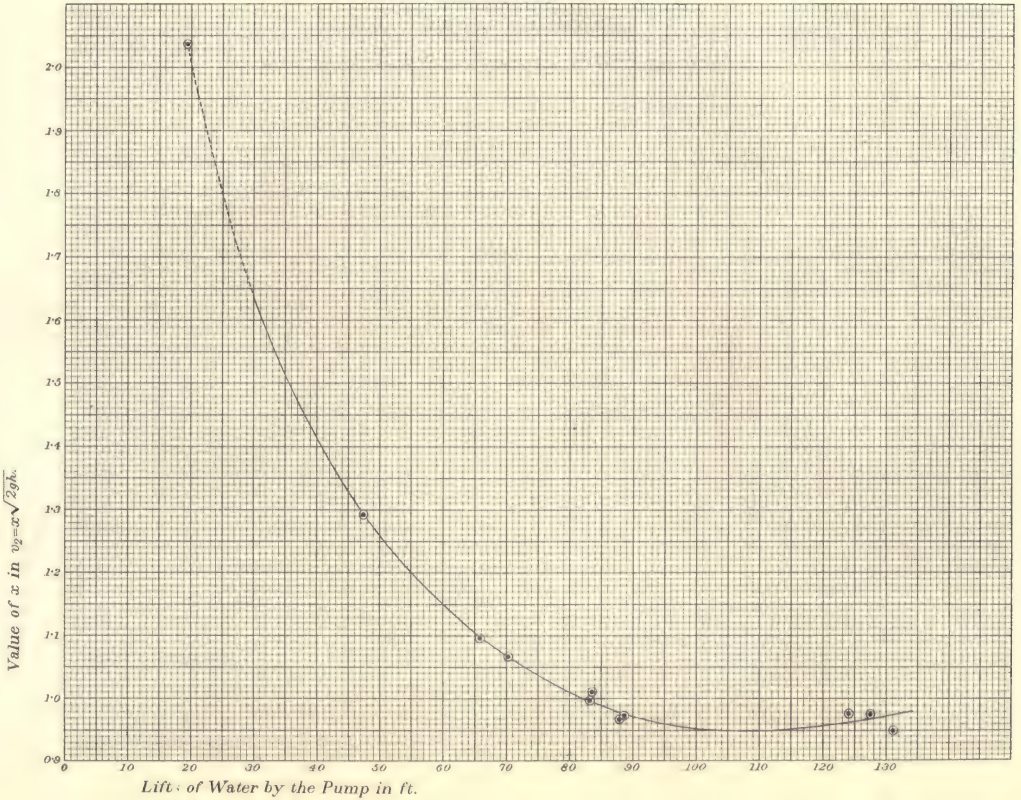
FIG. 12.



TESTS
OF
7″ FORCED VORTEX CENTRIFUGAL PUMP.

Showing the Coefficient x in the Expression for Peripheral Velocity of the Impeller $v_2 = x\sqrt{2gh}$ corresponding to Lift h up to 130 ft.

FIG. 13.



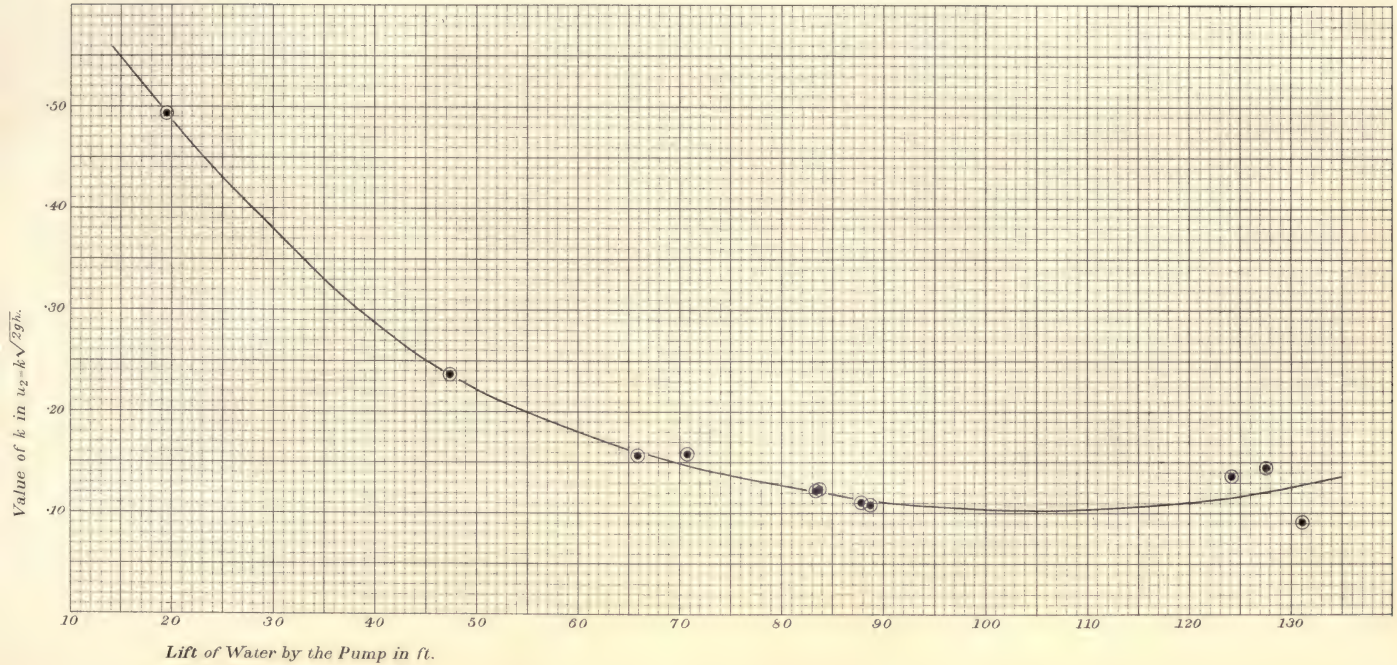
TESTS

OF

7″ FORCED VORTEX CENTRIFUGAL PUMP.

Showing the Coefficient k in the Expression for Radial Velocity of Outflow of Water $u_2 = k\sqrt{2gh}$ corresponding to Lift h up to 130 ft.

FIG. 14.



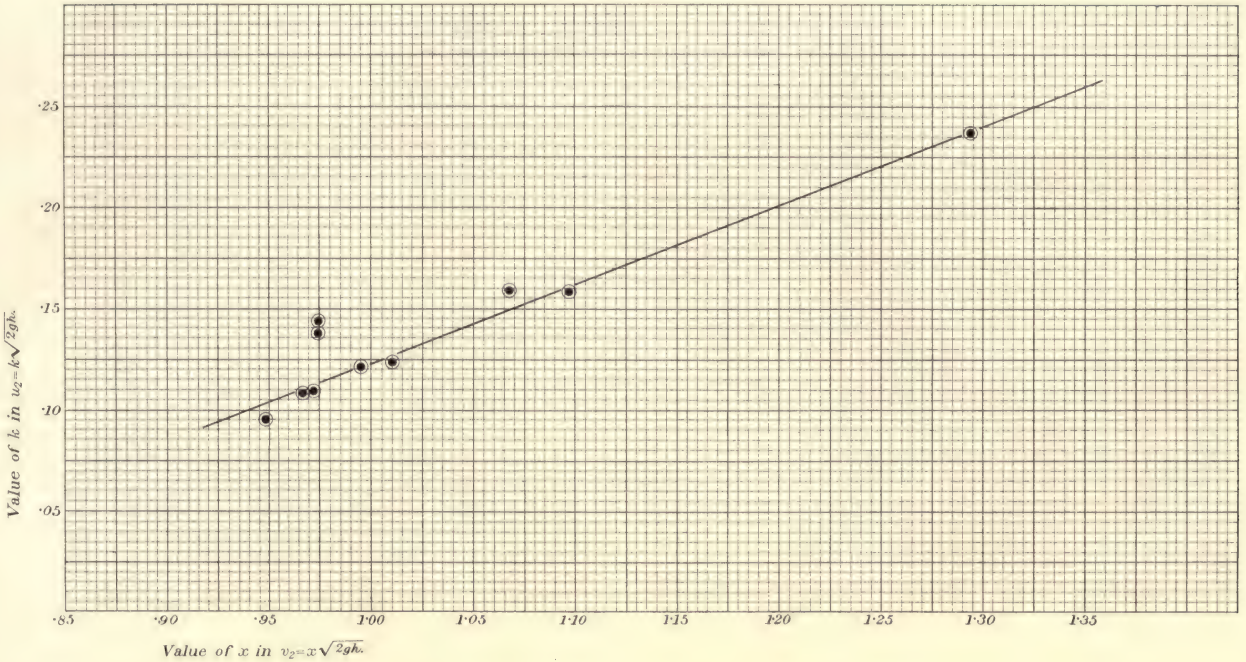
TESTS

OF

7″ FORCED VORTEX CENTRIFUGAL PUMP.

Showing the Relation between the Coefficient x in the Peripheral Velocity of the Impeller $v_2 = x\sqrt{2gh}$ and the Coefficient k in the Radial Velocity of Outflow of Water $u_2 = k\sqrt{2gh}$.

FIG. 15.



XXIV.

Formulæ for Helical Springs.

[*The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. VIII, No.* 14, *October,* 1905.]

An ordinary helical spring acted on by an axial force yields mainly by torsion. When the slope of the helix is very small, the bending action that accompanies the torsion is insignificant and the action of an axial load on the spring is practically equivalent to a twisting force applied at a radius of the helix to a straight shaft of a diameter equal to that of the wire of which the spring is made.

Helical Springs made of Round Steel Wires. Let D be the diameter of the helix measured to the centre lines of the coils of the helix; d the diameter of the wire of which the spring is made; n the number of coils; C the modulus of rigidity of the material; f the greatest safe shearing stress; W_1 the greatest safe axial load corresponding to f; δ_1 the extension or compression corresponding to W_1; W any axial load not exceeding the greatest safe load; δ the corresponding extension or compression. The load W_1 causing the spring to elongate or shorten, exerts a twisting moment $\frac{1}{2}DW_1$ on the wire and produces a shearing stress f. The moment of shearing stresses resisting twisting action is $\frac{\pi}{16}fd^3$.

Hence $$\frac{1}{2}DW_1=\frac{\pi}{16}f d^3,$$

$$W_1=\frac{\pi f d^3}{8D}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Now the angle of twist per unit length is equal to the twisting moment divided by the torsional rigidity. For a cylindrical bar of diameter d, the torsional rigidity is $C\frac{\pi}{32}d^4$ in which C is the modulus of rigidity of the material. If δ is the axial extension or compression of the helical spring due to a load W, the total angle of twist is $\frac{\delta}{\frac{1}{2}D}$. The length of the wire being $n\pi D$, the angle of twist per unit length is

$$\frac{\delta}{\frac{1}{2}n\pi D^2}.$$

Hence $$\frac{\delta}{\frac{1}{2}n\pi D^2}=\frac{\frac{1}{2}DW}{C\frac{\pi}{32}d^4},$$

$$\delta=\frac{\frac{1}{4}n\pi D^3 W}{C\frac{\pi}{32}d^4}$$

$$=\frac{8nD^3W}{Cd^4}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

in which both δ and W may have suffix 1. Combining (1) and (2), we have

$$\delta_1=\frac{\pi n f D^2}{Cd}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

Board of Trade rule for safety valve springs made of round steel wire is

$$W_1=\frac{8000\,d^3}{D}.$$

A rule for the extension or compression of the spring, given by D. K. Clark and often used in conjunction with the above rule, is

$$\frac{\delta}{n} = \frac{D^3 W}{22 \times (16\, d)^4}.$$

The safe shearing stress and the modulus of rigidity assumed in these rules are $f = 20{,}000$ lbs. per sq. inch and $C = 11{,}500{,}000$ lbs. per sq. inch. In Seatons's Pocket Book it is stated that the value adopted for the constant in (1), that is, for $\frac{1}{8}\pi f$ in Naval work is commonly 11,000 for round steel. This is equivalent to taking f equal to 28,000 lbs. per sq. inch. D. K. Clark's Mechanical Engineer's Pocket Book gives a formula for helical springs made of round steel,

$$W_1 = \frac{3 \times (16\, d)^3}{D},$$

which assumes the shearing stress f at 31,300 lbs. per sq. inch. Molesworth gives

$$W = \frac{2240 \times 3d^3}{\frac{1}{2}D},$$

which assumes f at 34,200 lbs. per sq. inch. Rankine's Useful Rules and Tables, page 304, gives the greatest safe stress f at 30,000 lbs. per sq. inch and the modulus of rigidity of steel and charcoal iron wire to range between 10,500,000 and 12,000,000 lbs. per sq. inch. Wilson Hartnell's experiments on helical springs used for governors and valves have shown that the safe limit of shearing stress is more than twice as great as that given by Rankine, being 60,000 to 70,000 lbs. per sq. inch for $\frac{3}{8}''$ wire and about 50,000 for $\frac{1}{2}''$ wire. The modulus of rigidity he found to vary from 12,000,000 to 13,700,000 lbs. per sq. inch for $\frac{1}{4}''$ wire; 11,000,000 for $\frac{11}{32}''$ wire; and from 10,600,000 to 10,900,000 for $\frac{3}{8}''$ wire. For $\frac{3}{8}''$ wire and under, he takes

$$W_1 = \frac{24000\, d^3}{D}.$$

The deflection due to any load W not exceeding the greatest safe load, he takes at

$$\delta = \frac{nD^3W}{1{,}440{,}000\, d^4}.$$

These correspond to f=61,100 lbs. per sq. inch and C=11,500,000 lbs. per sq. inch. The several values of the greatest safe shearing stress f given above may be thus compared:—

Board of Trade...	20,000 lbs. per sq. in.
Seaton...	28,000 „ „
Rankine	30,000 „ „
D. K. Clark	31,000 „ „
Molesworth...	34,000 „ „
Hartnell	61,000 „ „

Helical Springs made of Square Steel Wires. According to St. Venant the moment of shearing stresses in a shaft of square section is $\cdot 2082\, f h^3$, where h is the side of the square (See Prof. Cotterill's Ap. Mech., Appendix A.) Hence for a helical spring made of square wires, we have

$$W_1 = \frac{\cdot 4164\, f h^3}{D}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

The torsional rigidity of a square shaft according to St. Venant is $\cdot 14058\, Ch^4$. But the angle of twist per unit length of the wire in a helical spring is $\frac{\delta}{\frac{1}{2}n\pi D^2}$.

Hence

$$\frac{\delta}{\frac{1}{2}n\pi D^2} = \frac{\frac{1}{2}DW}{\cdot 14058\, Ch^4},$$

$$\delta = \frac{\frac{1}{4}n\pi D^3 W}{\cdot 14058 Ch^4} = \frac{nD^3W}{\cdot 1789 Ch^4}. \quad \ldots\ldots\ldots\ldots\ldots \quad (5)$$

Combining (4) and (5), we obtain

$$\delta = \frac{\cdot 7405 n \pi f D^2}{Ch}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

If Coulomb's hypothesis were true, the twisting moment of a square shaft would be equal to

$$T = \frac{\sqrt{2}}{6} f h^3$$

and

$$W_1 = \frac{\sqrt{2} f h^3}{3D}$$

$$= \frac{\cdot 4714 f h^3}{D}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4C)$$

which is 13·2 per cent. greater than the correct value, or in other words, ·8833 times this value is equal to the correct value of W_1 given by (4). Rankine gives in his Applied Mechanics, page 358,

$$T = \cdot 281 f h^3$$

as the result arrived at from St. Venant's theory applied to square shaft. Using this value of T we have

$$W_1 = \frac{\cdot 562 f h^3}{D}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4R)$$

This is certainly too large, being even greater than that given by Coulomb's fallacious theory. Suppose that for helical springs made of square steel wires Rankine's value of T is assumed and that the shearing stress f is taken at 20,000 lbs. /□″ as in Board of Trade Rules for helical springs of round steel wires, then the result is Board of Trade rule for helical springs made of square steel wires :—

$$W_1 = \frac{2 \times \cdot 281 \times 20{,}000\, h^3}{D}$$

$$= \frac{11{,}300\, h^3}{D}.$$

It thus appears that this rule of Board of Trade is based on Rankine's value of the moment of resistance to torsion, which is far too large. The correct value of the constant with $f = 20{,}000$ lbs. /□″ is 8,330 instead of 11,300. D. K. Clark gives in his Mechanical Engineer's Pocket Book an incorrect rule for helical springs made of square steel wire,

$$W_1 = \frac{(16)^3 \times 4 \cdot 29\, h^3}{D}.$$

The ratio of the constant in this formula to the constant in the previously given formula for the case of round steel wires is 1·43, which is exactly equal to the ratio $2 \times \cdot 281 \div \frac{\pi}{8}$, showing that the rule is from Rankine's formula. Since $2 \times \cdot 2082 \div \frac{\pi}{8} = 1 \cdot 060$, the correct value of the constant with $f = 31{,}300$ lbs. per sq. inch is $(16)^3 \times 1 \cdot 06 \times 3 = (16)^3 \times 3 \cdot 18$. If in any rule the ratio of the constant in (1) to that in (4) be greater than 1·06, that rule may be regarded as inaccurate, being derived either from Coulomb's hypothesis, or Rankine's formula, or some fallacious theory. In the rule given by Molesworth the said ratio is $4 \cdot 72 \div 3 = 1 \cdot 57$, which is certainly far too large. In Kempe's Mechanical Engineer's Year Book, the ratio is $\cdot 514 \div \frac{\pi}{8} = 1 \cdot 31$, which is also too large. In Seaton's Pocket Book, the formulæ (1) and (4) are put in the forms

$$W_1 = \frac{cd^3}{D} \quad \text{and} \quad W_1 = \frac{ch^3}{D},$$

and it is stated that the values of the constant c commonly adopted in Naval work are 11,000 and 15,000 for round and square steel respectively. Here the ratio is 1·31, which is again too large. If Coulomb's hypothesis were true, the extension or compression of the spiral springs made of square wires would be

$$\delta = \frac{\frac{1}{4}n\pi D^3 W}{\frac{1}{6}Ch^4} = \frac{nD^3 W}{\cdot 2122 Ch^4}, \quad \ldots\ldots\ldots\ldots \quad (5C)$$

which is ·84346 times the correct value of δ. The ratio of deflection of a helical spring made of square wire to the deflection of one made of round wire of a diameter equal to the side of the square is

$$\frac{\frac{1}{4}}{\cdot 14058} \div 8 = \cdot 6984.$$

This number may be useful in checking rules for deflection of helical springs made of round and square wires. Thus a rule often used in conjunction with Board of Trade rule and given by D. K. Clark is

$$\delta = \frac{nD^3 W}{a(16d)^4},$$

in which a is 22 for round steel and 30 for square steel. Here the constant a ought to be 22 and 31·4 respectively, if C is 11,500,000 lbs. per sq. inch; or else 21 and 30 respectively, if C is 11,000,000 lbs. per sq. inch. Seaton gives a equal to 26 for round steel and 32 for square steel. Here the value 26 for the round steel appears to be too large, as it makes C equal to about 13,600,000 lbs. per sq. inch.

Helical Springs made of Steel Wires of Rectangular Section. Prof. Reuleaux gives the following formulæ for helical springs made of wires of rectangular section :—

$$W = \frac{2f}{3D} \frac{b^2h^2}{\sqrt{b^2+h^2}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

and

$$\delta = \frac{3\pi n D^3 W}{4C} \frac{b^2+h^2}{b^3h^3}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

in which b and h are the breadth and depth of the section of the wire. The same formulæ are given in Prof. J. Perry's Applied Mechanics, Notes et Formules de l'Ingenieur revised and enlarged by L. A. Barré, Molesworth's Pocket Book, and some other books. It appears from Prof. Cotterill's remarks on shafts that the above formulæ have been derived from Cauchy's investigation of a shaft of rectangular section, prior to St. Venant's researches. For a wire of square section of side h, both the above formulæ reduce themselves to Coulomb's incorrect formulæ, namely, $(4C)$ and $(5C)$. As approximate empirical equations for helical springs made of wires of rectangular section, the right hand members of (7) and (8) may be multiplied by constants obtained by Saint Venant for shafts of rectangular section, thus :—

$$W = \frac{\cdot 8833 \times 2f}{3D} \frac{b^2h^2}{\sqrt{b^2+h^2}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

and

$$\delta = \frac{3\pi n D^3 W}{\cdot 8435 \times 4C} \frac{b^2+h^2}{b^3h^3}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

which, when b is equal to h, become the correct formulæ (4) and (5) for helical springs made of square wires. For the constants multipliers in these formulæ see Lord Kelvin's Natural Philosophy, Part II, Edition 1895, page 257.

Tokyo, August 11, 1905.

XXV.

Flexible Coupling.

[*The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. IX, No.* 15, *April,* 1906.]

Two shafts that are intended to be in one straight line are often slightly out of truth, so that they are neither parallel nor intersecting. Such shafts may be connected in a simple and effective manner by a new form of coupling shown in the annexed figures. This coupling was first designed by Prof. A. Inokuty, of the College of Engineering, Tokyo, and applied to a joint of a shafting which was a constant source of trouble in a large paper mill. The alignment of the said shafting $7\frac{7}{16}''$ in diameter could be adjusted approximately true when transmitting the full power of nearly 2000 horse power; but at a much reduced power the discrepancy of the centres sometimes amounted to even more than half an inch. Quite satisfactory was the result of application of the new coupling.

It will be seen from the figures that Prof. A. Inokuty's coupling is a very simple but clever modification of Oldham coupling. It consists of three pieces, of which the outer two have, each, slightly curved recesses lying in one axial plane and which are fitted one on each shaft like the two parts of a claw coupling. The middle piece which slides between the outer two is a disc

with its central part removed and is provided on opposite faces with straight diametral projections or teeth at right angles to one another. These teeth fit closely on their lateral faces into the corresponding recesses in the two outer pieces. The sections of the recesses made by axial planes of the shaft are somewhat convex and the depth of the recesses is a little less than the height of the corresponding teeth. By this construction some degree of flexibility is given to the coupling, besides the adaptability to any small dislocation of the two shafts. The following rules given by Prof. A. Inokuty may be used in designing this coupling, the parts being of cast iron :—

D = diameter of shaft,

For shafts up to 4″ dia.

$$F_0 = F_1 - \tfrac{1}{2}''$$
$$F_2 = 3D + 2''$$
$$T = \cdot 3D + \tfrac{1}{2}''$$
$$R = D + \tfrac{2}{3}''$$

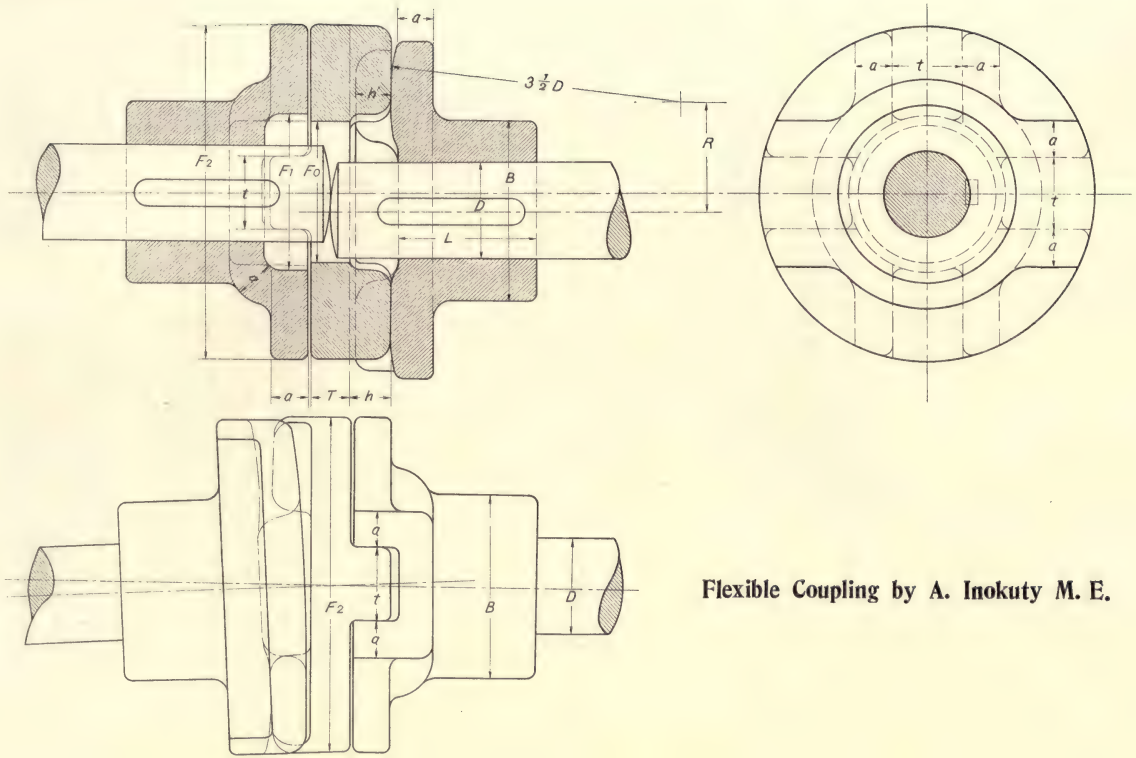
For shafts above 4″ dia.

$$F_0 = 1 \cdot 62D$$
$$F_2 = 3\tfrac{1}{2}D$$
$$T = \cdot 42D$$
$$R = 1\tfrac{1}{6}D$$

For shafts of any dia.

$$F_1 = \tfrac{1}{2}F_2$$
$$h = \cdot 42D$$
$$h' = \cdot 40D - \cdot 08''$$
$$t = \cdot 75D$$
$$a = \tfrac{1}{3}(D + 1'')$$
$$B = 1 \cdot 8D + \cdot 8''$$
$$L = 1 \cdot 4D + \tfrac{1}{2}''$$

The bending stress induced at the roots of the teeth on the middle piece is not greater than $0 \cdot 38f$, f being the maximum shearing stress due to the twisting moment transmitted by the shafting.



Flexible Coupling by A. Inokuty M. E.

XXVI.

Similar Structures with Corresponding Loads.

[*The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. IX, No.* 17, *September*, 1907.]

Suppose that several structures are made from the same drawing on different scales and loaded in the same manner, the materials of which the corresponding parts are composed being also the same. The sizes of the structures are supposed to be such that in estimating the strength of the parts the stresses due to their own weights have to be taken into consideration along with the stresses due to the externally applied loads. These are similar structures and may be said to be under the action of *corresponding loads*. The weight of such a structure itself varies evidently as the cube of the linear dimensions while its power to sustain a load varies in majority of cases as the square of the dimensions. Hence in similar structures there exists generally a certain definite size for which the externally applied load is a maximum.

As the structure becomes larger beyond that size, its power to resist an external force gradually diminishes till a limiting size is reached for which the structure is just capable of sup-

porting its own weight only. A structure larger than this can not exist. A common housefly can walk on a wall or on a ceiling with its back down; but a fly of the size of an elephant, if it ever existed, would not be able to do so.

Let W = the external load applied on a structure in a specific manner,

W_0 = the weight of the structure itself,

p = the intensity of stress on a particular member due to W,

p_0 = the intensity of stress on the same member due to W_0,

l = the length of the same member, or of any other particular member; for girders, roofs, bridges, &c. l may conveniently be taken as the span,

A = the sectional area of the member whose length is l,

f = the safe working resultant stress.

Then
$$p = \text{const.} \times \frac{W}{A}$$
$$= k\frac{W}{l^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$
$$p_0 = \text{const.} \times \frac{W_0}{A}$$
$$= k_0\frac{W_0}{l^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

and
$$W_0 = c\,l^3. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

In these equations k is a constant coefficient, being an abstract number depending on the mode of distribution of the external load W upon the structure; k_0 is the value of k corresponding to the weight of the structure W_0; c is also a constant

coefficient expressed in units of weight per unit volume. Now it being assumed that p and p_0 are stresses at a point in the same section, acting in the same line of action, we have

$$f=p+p_0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Substituting in this the values of p and p_0 given by (1) and (2), we have

$$f l^2=kW+k_0W_0,$$

which by (3) becomes

$$f l^2=kW+ck_0l^3.$$

Hence $$W=\frac{1}{k}(f-c\,k_0l)l^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

The external load W which a structure can safely bear therefore becomes a maximum, when the length l is

$$l=\frac{2}{3}\cdot\frac{f}{c\,k_0}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

and

$$\textit{Maximum}\quad W=\frac{fl^2}{3k}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

The limiting size for which a structure can support its own weight only is obtained by equating the right hand member of (5) to zero,

thus :— $$l=\frac{f}{c\,k_0},$$

which is thus seen to be $1\frac{1}{2}$ times the size for maximum external load. These results may be illustrated by a few simple examples.

Example 1. Consider a beam of uniform rectangular section of breadth b, depth h, and span l, supported at the ends and loaded at the centre of the span.

Equating the maximum bending moment at the centre of the span to the moment of resistance of the section, we have

$$\frac{1}{4}Wl+\frac{1}{8}w_0bhl^2=\frac{1}{6}fbh^2,$$

$$W=\frac{2}{3}f\frac{bh^2}{l}-\frac{1}{2}w_0bhl$$

$$=\frac{2}{3}f\left(\frac{bh^2}{l^3}\right)l^2-\frac{1}{2}w_0\left(\frac{bh}{l^2}\right)l^3.$$

Therefore $$\frac{dW}{dl}=\frac{4}{3}f\left(\frac{bh^2}{l^3}\right)l-\frac{3}{2}w_0\left(\frac{bh}{l^2}\right)l^2.$$

Equating this to zero we obtain the span l corresponding to the maximum load W on the beam,

$$l=\frac{8fh}{9w_0l}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

and

$$\textit{maximum}\quad W=\frac{2fbh^2}{9l}=\frac{2f}{9}\left(\frac{8f}{9w_0}\right)^2\left(\frac{b}{l}\right)\left(\frac{h}{l}\right)^4, \quad \ldots\ldots \quad (9)$$

in which w_0 is the weight of the material of the beam per cubic unit. Suppose that the beam is of cast iron and has a depth equal to one 16th of the span. Taking the safe bending stress f at 2000 lbs. per sq. inch and the weight w_0 per cub. inch at 0·27 lbs., we have

$$l=\frac{8fh}{9w_0l}=\frac{8\times2000\times1}{9\times0{\cdot}27\times16}=411{\cdot}5''$$

$$=34{\cdot}3 \text{ ft.}$$

Taking the breadth of the beam equal to $b=0{\cdot}017$ times the span, we have

$$\textit{maximum}\quad W=\frac{2fbh^2}{9l}=\frac{2f}{9}\left(\frac{8f}{9w_0}\right)^2\left(\frac{b}{l}\right)\left(\frac{h}{l}\right)^4$$

$$=\frac{2\times 2000}{9}\left(\frac{8\times 2000}{9\times{\cdot}27}\right)^2({\cdot}017)\left(\frac{1}{16}\right)^4$$

$$=5{,}000\text{ lbs.}=2{\cdot}23\text{ tons.}$$

It is worth while to note that for a simple rectangular beam of large span used as in this example, timber would be much superior to cast iron. This follows at once from (8), since the l depends on the ratio of f to w_0. Thus for *keyaki** the coefficient of breaking bending strength is about 14,700 lbs. per sq. inch and weight of this timber is 52 lbs. per cubic ft. Taking the safe bending stress f at 1000 lbs. per sq. inch, we obtain the span l corresponding to the maximum load equal to 4·48 times that for cast iron beam. The maximum load which the beam can support at the centre of the span is then 10·0 times that for cast iron beam.

A well-known analogous fact relating to a flywheel may here be mentioned. The kinetic energy stored in the rim of a flywheel is equal to $\frac{1}{2}fV$, in which f is the safe tensile stress induced in the material of the rim by centrifugal force and V the volume of the rim. This expression may be written in the form

$$\frac{f}{2w_0}W,$$

in which W is the total weight of the rim. Hence the store of kinetic energy per lb. of the rim is very much greater when the flywheel is made of wood than when of cast iron.

* A Japanese timber.

Example 2. Let a solid cylindrical column of horizontal sectional area A and height h be loaded at the top by a weight W.

The compressive stress f at the bottom of the column is given by

$$fA = w_0 Ah + W,$$

$$W = fA - w_0 Ah$$

$$= f\left(\frac{A}{h^2}\right)h^2 - w_0\left(\frac{A}{h^2}\right)h^3.$$

$$\frac{dW}{dh} = 2f\left(\frac{A}{h^2}\right)h - 3w_0\left(\frac{A}{h^2}\right)h^2.$$

Equating this to zero, we obtain the height for the maximum load,

$$h = \frac{2f}{3w_0},$$

in which w_0 is the weight of the material of the column per cubic unit. The maximum load is $\frac{1}{3}fA$.

Suppose that the column is of brickwork weighing 0·062 lb. per cub. inch, with the safe compressive stress equal to 120 lbs. per sq. inch. The height of the column for maximum load is then

$$h = \frac{2f}{3w_0} = \frac{2 \times 120}{3 \times 0{\cdot}062} = 107{\cdot}5 \text{ ft.}$$

and the limiting height for which the column can support itself without a load on the top is only 161 ft.

Example 3. A steel box girder consists of the following part riveted together by rivets $\frac{13''}{16}$ dia. (after riveting):— $42'' \times \frac{1''}{2}$ web plates, $30'' \times \frac{11''}{16}$ flange plates, and $5'' \times 4'' \times \frac{1''}{2}$ angle bars. The weight of the girder is 30·00 lbs. per inch length of the

girder and the moment of inertia of the section about an axis passing through the center of gravity of the section and parallel to the flanges is equal to 26,990 biquad. inches, so that considering 42″ as the depth of the beam, the modulus of the section is $Z=1285$ cub. inches. Now let it be assumed that box girders similar in section to the one just given are employed for different spans which bear a constant ratio to the depths of the girders. The weight of such girders in lbs. per inch length is given by $w=0{\cdot}0170h^2$ and the modulus of the section Z in cubic inches by $0{\cdot}0174h^3$, h being the depth of the web plate in inches.

Suppose that the girder is supported at the ends and loaded by a total weight W distributed uniformly over the span. Equating the greatest bending moment at the centre of the span to the moment of resistance of the section, we have

$$\frac{1}{8}Wl+\frac{1}{8}wl^2=fZ,$$

$$W=\frac{8fZ}{l}-wl$$

$$=\frac{8f}{l}\times 0{\cdot}0174\,h^3-0{\cdot}0170\,h^2l$$

$$=8\times 0{\cdot}0174f\left(\frac{h}{l}\right)^3l^2-0{\cdot}0170\left(\frac{h}{l}\right)^2l^2.$$

$$\frac{dW}{dl}=16\times 0{\cdot}0174f\left(\frac{h}{l}\right)^3l-3\times 0{\cdot}0170\left(\frac{h}{l}\right)^2l^2.$$

Equating this to zero, we obtain the span l for maximum load,

$$l=\frac{16f}{3}\cdot\frac{0{\cdot}0174}{0{\cdot}0170}\cdot\frac{h}{l}.$$

If f be taken at 12,000 lbs. per sq. inch and the depth h equal to one 20th of the span, the span for greatest load is

$$l = \frac{16 \times 12000}{3} \times \frac{0{\cdot}0174}{0{\cdot}0170} \times \frac{1}{20} \text{ inches}$$

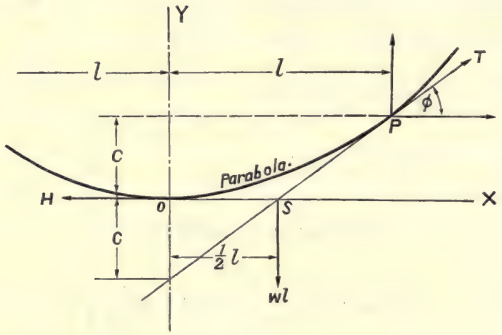
$$= 273 \text{ ft.}$$

Maximum $$W = \frac{1}{2} \times 0{\cdot}0170 \left(\frac{16f}{3} \cdot \frac{0{\cdot}0174}{0{\cdot}0170} \right)^3 \left(\frac{h}{l} \right)^5$$

$$= 740{,}000 \text{ lbs.} = 330 \text{ tons,}$$

being very nearly 1·2 tons per foot of the span.

Example 4. Let a cable suspended from two points in one level line bear a distributed vertical load which is uniform per unit of horizontal distance. Assume that the girth of the cable and the dip of the suspension curve have each a definite ratio to the span.



Let $2l$=the span,

c=the dip or sag of suspension cable.

w=resultant vertical load on the cable estimated per unit of horizontal distance, (including the weight of the cable itself),

W=the total external vertical load which is uniform per unit of horizontal distance,

w_0=the weight of the material of the cable per cubic unit,

f=the safe working tensile stress of the cable,

ϕ=the inclination of the cable to the horizon at each point of support,

T=the tension of the cable at each point of support,

H=horizontal tension at the lowest point

=horizontal component of tension at any point.

Then the equation of the parabola to which the suspension curve approximates is

$$y=\frac{wx^2}{2H}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

in which H is given by

$$H=\frac{wl^2}{2c}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

The tension at the point of support P is

$$T=\sqrt{H^2+(wl)^2}=\sqrt{\left\{\frac{wl^2}{2c}\right\}^2+(wl)^2}$$

$$=wl\sqrt{\left\{\frac{l}{2c}\right\}^2+1}=wl\operatorname{cosec}\phi. \quad \ldots\ldots\ldots\ldots \quad (3)$$

Now $T=fA$ and $w=w_0A+\frac{W}{2l}$.

Substituting these in (3) we obtain

$$W=2fA\sin\phi-2w_0Al$$

$$=2f\left(\frac{A}{l^2}\right)l^2\sin\phi-2w_0\left(\frac{A}{l^2}\right)l^3.$$

$$\frac{dW}{dl}=4f\left(\frac{A}{l^2}\right)l\ \sin\phi-6w_0\left(\frac{A}{l^2}\right)l^2.$$

Equating this to zero, we obtain the span $2l$ for maximum load,

$$2l = \frac{4f \sin \phi}{3w_0},$$

and

$$\textit{maximum}\quad W = \frac{2}{3} f A \sin \phi.$$

Suppose that the dip $c = \frac{1}{20}$span, $f = 12$ tons per sq. inch, $w_0 = 0{\cdot}283$ lb. per cub. inch. Then $\sin \phi = (2c)/\sqrt{l^2 + (2c)^2}$ $= 2 \times \frac{1}{10} \div \sqrt{1 + 4 \times \frac{1}{100}} = 0{\cdot}1961$ and

$$2l = \frac{4 \times 2240 \times 12}{3 \times 0{\cdot}283} \times 0{\cdot}1961 = 24{,}840''$$

$$= 2{,}070 \text{ ft.}$$

The height of each pier above the lowest point (equal to the dip) is $103\frac{1}{2}$ ft.

Let us now consider an analogous problem relating to **stresses brought into play in similar machines.** The stress at a point in a section of a machine working at a steady speed is made up of two parts; one part is the stress due to the useful working force transmitted through the section and the other that due to the accelerating force arising from the reciprocation or rotation of the mass of machine parts. Belts and ropes transmitting power at a high speed, the valve spindles and connecting rods of a high speed engine, the coupling rods of a lecomotive engine, &c. are examples of machine parts in which both these stresses have to be taken into account. In a high speed flywheel rotating uniformly the stress induced at any part of it is that due to the rotation only.

Let W=the working load applied on a machine in a specific manner,

n=the number of revolutions of the machine per unit time,

p=the intensity of stress on a particular part of the machine due to W,

p_0=the intensity of stress on the same member due to the accelerating force, and

l, A, k, and f have the same meanings as in the previous case of large structures.

The accelerating force is proportional to $l \times \text{mass} \times n^2$ and therefore the stress due to it is

$$p_0 = \text{const.} \times \frac{l \times \text{mass} \times n^2}{A}.$$

But $$\text{the mass} = \frac{1}{g} w_0 l A,$$

in which w_0 is the weight of the material of the machine part per cubic unit. Hence we have

$$p_0 = \text{const.} \frac{w_0}{g} n^2 l^2 = a n^2 l^2, \quad \ldots\ldots\ldots\ldots\ldots \quad (10)$$

in which a is a constant for different sizes and speeds of the machine. The safe working resultant stress is

$$f = k \frac{W}{l^2} + a n^2 l^2, \quad \ldots\ldots\ldots\ldots\ldots \quad (11)$$

from which we obtain

$$\frac{dW}{dl} = \frac{1}{k}(2fl - 4a n^2 l^3).$$

The working load on the machine therefore becomes a maximum when the length l is given by

$$\frac{1}{2} f = a n^2 l^2 \quad \ldots\ldots\ldots\ldots\ldots \quad (12)$$

and the corresponding load is

$$maximum\ W=\frac{1}{2}\cdot\frac{fl^2}{k}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (13)$$

It is thus seen that when the working load on a machine is maximum, the two parts of the stress above mentioned are each one half of the safe working stress. For example if a coupling rod of uniform rectangular section of breadth b and depth h is jointed at each end to a crank of radius r rotating uniformly at n revolutions per second, the greatest bending stress occurring at the centre of the length l is given by

$$\frac{1}{2}f=\frac{3\,w_0}{4g}\cdot\frac{r}{h}\cdot(2\pi nl)^2,$$

in which f is the safe working stress and w_0 the weight per cubic unit of the material of the rod.

The greatest thrust or pull that can be transmitted through the rod is $W=\frac{1}{2}fbh$.

In some machine parts the working load W is nothing and in that case supposing n to be constant, the limiting size for which a machine can stand against its own accelerating force only is obtained by equating W in the equation (11) to zero; thus :—

$$f=a\,n^2\,l^2,$$

or in other words, the limiting linear speed is constant for a machine of any size. For the rim of a flywheel this formula becomes

$$f=\frac{w}{g}v^2,$$

in which f is the tensile stress in the rim and v the linear velocity. For a flywheel in the form of a circular disk of uni-

form thickness, having a concentric hole, the circumferential tensile stress is greatest at the inner surface of the hole and is given by

$$f=\frac{w}{4g}\left\{\left(3+\frac{1}{m}\right)+\left(1-\frac{1}{m}\right)\frac{r_0^2}{r^2}\right\}v^2,$$

in which r and r_0 are the outer and inner radii of the disk respectively, v the linear velocity of the outer circumference of the disk, and m the reciprocal of Poisson's ratio for the material of the flywheel. We have cited enough examples illustrating the fact relating to large similar structures and high speed machines.

XXVII.

Column of Uniform Strength.

[*Journal of the Engineering Society, Japan, Nos.* 300, 301, 302, 1907.]

A long straight column bearing an axial compressive load undergoes a lateral deflection when that load exceeds a certain value, and the deflected axis assumes a sine curve if the column is homogeneous and of uniform section and the stress nowhere exceeds the elastic limit. The bending stress thus induced in the material of the column varies from one section to another along the length, being greatest at a section furthest from the original straight axis and zero at a point of inflection. Now the lateral dimensions of a long straight column can be so varied as to make the greatest bending stress for a section constant for the whole length. Such a column, being analogous to a beam of so-called uniform strength, may be termed a column of uniform strength. Formulæ for the strength of columns of this form can be obtained theoretically by considering the bending action alone and the result is almost the same as Euler formula, the only difference being in the value of the coefficients. For columns of moderate length such as occur in practice, it is necessary to consider the direct compressive stress as well as the bending stress due to the deflection and for this purpose the

writer has obtained a set of semi-empirical formulæ in the form of Gorden-Rankine formula. These formulæ might be useful in designing large columns of approximately uniform strength as for example, shear legs, connecting rods for large engines, &c.

Referring to Fig. 1, let a straight bar of great length be acted on by axial forces P, P, tending to compress the bar along

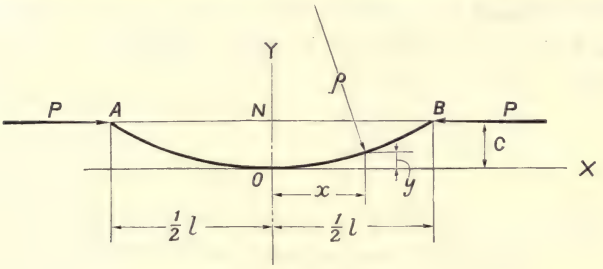


FIG. 1.

the original straight axis ANB, and let it be bent into a curve AOB, O being the centre of the length and $ON=c$ the maximum deflection at the centre. Since this deflection c is generally a very small quantity compared with the length of the bar l it is accurate enough to assume NA and NB each equal to half the length. Take the tangent at O to the curve of deflection for the axis of x and the normal at O for that of y. At any section indicated by (x, y) the bending moment is

$$M=(c-y)P. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

The relation between the bending moment and the curvature of the bar is

$$\frac{1}{\rho}=\frac{d^2y}{dx^2}=\frac{(c-y)P}{EI}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

and the equation for the strength is

$$(c-y)P=f\frac{I}{y'}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

in which y' is the distance of the part of the bar furthest from the neutral axis of the section on the tension or compression side, ρ the radius of curvature at the point (x, y), and I the geometrical moment of inertia of the section about its neutral axis perpendicular to the plane of bending. Combining (2) and (3) we obtain

$$\frac{1}{\rho}=\frac{d^2y}{dx^2}=\frac{f}{Ey'}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Case I. *A long straight column of uniform strength, rounded at both ends, having a rectangular section with the sides parallel to the plane of bending constant.*

Fig. 2 represents the front and side view of the column of length l under the action of compressive load P, the width of the section at a distance x from the centre being b and the constant depth h as shown in the figure. Putting $y'=\frac{1}{2}h$ in (4), we obtain

$$\frac{1}{\rho}=\frac{2f}{Eh}.$$

The greatest bending stress f is to remain constant for any section and therefore it is seen that the column bends under the load P into an arc of a circle of radius ρ given by the above relation. If I_0 is the value of I at the centre, we have from (2)

$$P=\frac{EI_0}{c\rho}.$$

Now from a well known property of a circle we have the following relation between a chord and the corresponding versed sine,

$$c(2\rho - c) = (\tfrac{1}{2}l)^2.$$

The deflection c at the centre is a very small quantity and may be neglected in comparison with the radius ρ; we have then

$$c\rho = \tfrac{1}{8}l^2. \quad \ldots\ldots\ldots\ldots \quad (5)$$

Substituting this in the expression for P above given, we obtain

$$P = 8\frac{EI_0}{l^2}, \quad \ldots\ldots\ldots \quad (A_1)$$

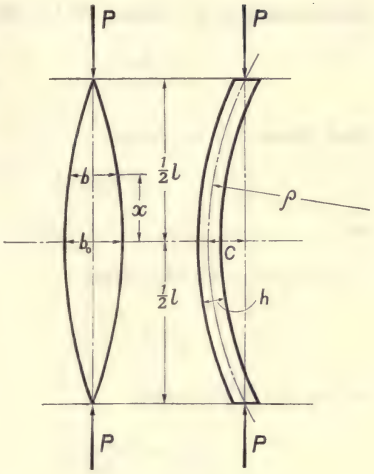


FIG. 2.

which is of the same form as Euler formula for a long column of uniform section rounded at both ends; the only difference is in the coefficient, which in the present result is 8, while in Euler's it is π^2. It may be remarked that in a long column of uniform section if a load P satisfies the condition of equilibrium expressed by Euler formula, any deflected position is a position of equilibrium under the same load P and consequently the deflection c is an indeterminate quantity. The same thing happens also in the case of a column of uniform strength. Now to find the variable width b of the column, the well known property of a circle above cited may again be made use of; thus

$$(2\rho - y)y = x^2,$$

in which y may safely be neglected in comparison with 2ρ. Then

$$y = \frac{x^2}{2\rho}.$$

Eliminating ρ by means of (5) we have

$$y=\frac{4cx^2}{l^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

Now from (3) we have

$$(c-y)P=\tfrac{1}{6}fbh^2$$

and

$$cP=\tfrac{1}{6}fb_0h^2.$$

Dividing one by the other,

$$\frac{b}{b_0}=1-\frac{y}{c},$$

which by (6) becomes

$$\frac{b}{b_0}=1-\left(\frac{x}{\frac{1}{2}l}\right)^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

This is a parabola with its axis at right angles to the length of the column and passing through the centre. Each of two parabolic arcs forming the outline of the column may be approximated by two straight lines symmetrically sloping towards the ends of the column and touching the parabolic arc. Of all such sets of straight lines, that set which requires the least amount of material for the column is the following:—Set off a width equal to $1\frac{1}{4}b_0$ at the centre and a width equal to $\frac{1}{4}b_0$ at each end of the column. Join the ends of these widths by straight lines, which will touch the parabolic arcs at $x=\frac{1}{4}l$, the width at that section being equal to $\frac{3}{4}b_0$.

The result of this Case I for columns of rectangular section may be easily extended to columns of any section, such as hollow rectangle, symmetrical double the section, cruciform section, &c. The constant dimension h must then be regarded as the sides parallel to the plane of bending, of a circumscribing rectangle.

Case II. *A long straight column of uniform strength, rounded at both ends, the dimension of the section perpendicular to the plane of bending being constant.*

Referring to Fig. 3, the width b of a section is now constant while the depth h is variable, so that the depth at the centre is distinguished by h_0. The variable moment of inertia of a section at distance x can be written as follows:—

$$I=\frac{h^3}{h_0^3}I_0.$$

By substitution in (2) we obtain

$$\frac{d^2y}{dx^2}=\frac{(c-y)P}{EI_0}\left(\frac{h_0}{h}\right)^3.$$

On making x and y equal to zero simultaneously, this becomes

$$\frac{1}{\rho_0}=\frac{cP}{EI_0}, \quad \ldots\ldots\ldots\ldots \quad (8)$$

in which ρ_0 is the radius of curvature of the elastic curve assumed by the axis of the column, at the centre of the length. The above equation for the second differential coefficient of y may then be written in the form

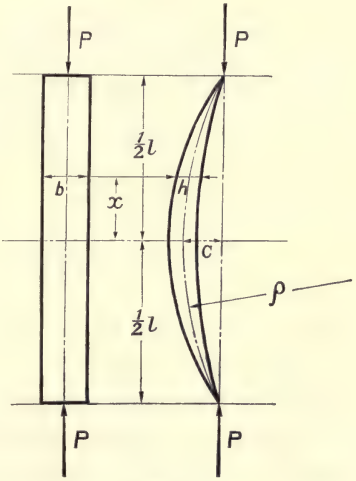


Fig. 3.

$$\frac{d^2y}{dx^2}=\frac{c-y}{c\rho_0}\left(\frac{h_0}{h}\right)^3. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

Substituting the value of I in (3) we have

$$(c-y)P=\frac{2fI_0}{h_0}\left(\frac{h}{h_0}\right)^2.$$

On making x and y equal to zero simultaneously, this becomes

$$cP=\frac{2fI_0}{h_0},$$

which being substituted in the above equation gives

$$\frac{c}{c-y}=\left(\frac{h_0}{h}\right)^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

Combining (9) and (10) we obtain

$$\frac{d^2y}{dx^2}=\frac{1}{\rho_0}\frac{c^{\frac{1}{2}}}{(c-y)^{\frac{1}{2}}}.$$

Integrating and noting that dy/dx is zero at the origin, we have

$$\left(\frac{dy}{dx}\right)^2=\frac{4c^{\frac{1}{2}}}{\rho_0}\left\{c^{\frac{1}{2}}-(c-y)^{\frac{1}{2}}\right\}.$$

Integrating again we obtain

$$\frac{2}{3}\left\{c^{\frac{1}{2}}-(c-y)^{\frac{1}{2}}\right\}^{\frac{1}{2}}\left\{2c^{\frac{1}{2}}+(c-y)^{\frac{1}{2}}\right\}=\frac{c^{\frac{1}{2}}}{\rho_0^{\frac{1}{2}}}x. \quad \ldots\ldots\ldots\ldots \quad (11)$$

This is the equation of the elastic curve assumed by the axis of the column in Case II. Now c is the value of y when x is equal to $\frac{1}{2}l$. Substituting these values in (11), there is obtained an equation of condition that the column may deflect under a load P and keep an equilibrium in a curved form expressed by (11),

$$c\rho_0=\frac{9}{64}l^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

Eliminating $c\rho_0$ between this and (8), there results

$$P=\frac{64}{9}\frac{EI_0}{l^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (A_2)$$

which is again of the same form as Euler formula. The coefficient $\frac{64}{9}$ is equal to 0·721 times π^2, so that the strength of a

column having the form stated in Case II is 72 per cent. of the strength of a column of uniform section and of the same material, the form and the area of the section at the centre being the same.

Eliminate y and ρ_0 from (11) by means of (10) and (12); we then obtain

$$\frac{x}{\frac{1}{2}l} = \frac{1}{2}\left(1-\frac{h}{h_0}\right)^{\frac{1}{2}}\left(2+\frac{h}{h_0}\right). \quad \ldots\ldots\ldots\ldots \quad (13)$$

This is the equation of the curve which the outline of the column must have in order that it may be of uniform strength. The form of this curve is somewhat like a blunt ellipse; it can be drawn easily by setting off a few values of $x/\frac{1}{2}l$ and the corresponding values of h/h_0.

$\frac{h}{h_0}$	0·95	0·90	0·75	0·50	0·25
$\frac{x}{\frac{1}{2}l}$	0·330	0·459	0·688	0·884	0·974

The curved outline of the column may be approximated by two pairs of straight lines touching the curve and sloping towards the ends of the column symmetrically about the cross section at the centre. Of all tangents that can be drawn thus, those which require the least amount of material for the column are the following:— At the centre of the length set off a width equal to $1{\cdot}143h_0$ and at each end set off a width equal to $0{\cdot}616h_0$. Straight lines joining the ends of these widths form the tangents sought. These touch the curve (13) at a distance $\frac{1}{4}l$ from the centre and the value of h at that section is given by the equation

$$h^3+3h_0h^2-3h_0^3=0,$$

which by solution gives $h=0{\cdot}879\,h_0$. The widths above given have been calculated by

$$(0{\cdot}8794+\tfrac{1}{2}m)h_0$$

and $$(0{\cdot}8794-\tfrac{1}{2}m)h_0,$$

in which

$$m=\frac{4\times(1-0{\cdot}8794)^3}{3\times 0{\cdot}8794}h_0$$

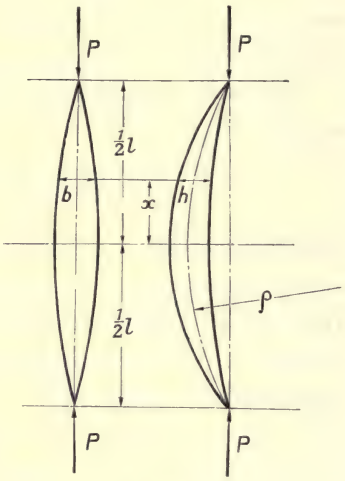


Fig. 4.

Case III. *A long straight column of uniform strength, with both ends rounded, the cross sections being similar figures.*

The variable moment of inertia of a section distant x from the centre may be written

$$I=\left(\frac{h}{h_0}\right)^4 I_0.$$

Substituting this in (2) we have

$$\frac{d^2y}{dx^2}=\frac{(c-y)P}{EI_0}\left(\frac{h_0}{h}\right)^4.$$

When $x=0$, $y=0$ and $h=h_0$ and therefore

$$\frac{1}{\rho_0}=\frac{cP}{EI_0}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

Hence we obtain

$$\frac{d^2y}{dx^2}=\frac{c-y}{c\rho_0}\left(\frac{h_0}{h}\right)^4. \qquad \ldots\ldots\ldots\ldots\ldots \quad (15)$$

Substituting in (3) the value of I we have

$$(c-y)P=\frac{2fI_0}{h_0}\left(\frac{h}{h_0}\right)^3.$$

When $x=0$, $y=0$ and $h=h_0$ and therefore

$$cP=\frac{2fI_0}{h_0}.$$

Hence we obtain

$$\frac{c-y}{c}=\left(\frac{h}{h_0}\right)^3. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (16)$$

Eliminating h between (15) and (16) we have

$$\frac{d^2y}{dx^2}=\frac{1}{\rho_0}\frac{c^{\frac{1}{3}}}{(c-y)^{\frac{1}{3}}}.$$

Integrating and observing that $dy/dx=0$ when $y=0$, we obtain

$$\left(\frac{dy}{dx}\right)^2=9A^2\left\{c^{\frac{2}{3}}-(c-y)^{\frac{2}{3}}\right\},$$

in which $3A^2=c^{\frac{1}{3}}/\rho_0$. Now writing $z^3=c-y$ and $a^3=c$, the equation becomes

$$\left\{(a^2-z^2)^{\frac{1}{2}}-\frac{a^2}{(a^2-z^2)^{\frac{1}{2}}}\right\}dz=Adx.$$

Integrating and observing that when $x=0$, y is also nothing and z becomes equal to $c^{\frac{1}{3}}=a$, we obtain

$$\tfrac{1}{2}z\sqrt{a^2-z^2}+\tfrac{1}{2}a^2\cos^{-1}\frac{z}{a}=Ax, \quad \ldots\ldots\ldots \quad (17)$$

which is the equation of the elastic curve assumed by the axis of the column in Case III. Now c is the value of y when $x=\frac{1}{2}l$, that is, when $z=0$. Substitution of this relation between the constants in (17) gives an equation of condition that the column may deflect under a load P and keep an equilibrium in a curved form expressed by the equation (17), thus

$$c\rho_0 = \frac{4}{3\pi^2} l^2.$$

Substituting this in (14) we obtain

$$P = \frac{3}{4} \pi^2 \frac{EI_0}{l^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (A_3)$$

which is again of the same form as Euler formula. A column of uniform strength having cross sections similar, is therefore $\frac{3}{4}$ times as strong as a column of uniform section made of the same material, the form and dimensions of the section at the centre being the same.

The equation (16) can be witten in the form

$$\frac{z}{a} = \frac{h}{h_0}.$$

Substituting this value of z and the value of ρ_0 above found in the equation (17), we obtain

$$\frac{\pi}{2} \frac{x}{\frac{1}{2}l} = \frac{h}{h_0} \sqrt{1 - \frac{h^2}{h_0^2}} + \cos^{-1} \frac{h}{h_0}, \quad \ldots\ldots \quad (18)$$

which is the equation of the curve to be given to the outline of the column in Case III in order that it may be of uniform strength. Assume that

$$h = h_0 \cos \tfrac{1}{2}\theta,$$

then the above equation becomes

$$\frac{x}{\frac{1}{2}l} = \frac{1}{\pi}(\sin\theta + \theta).$$

The curved outline of the column can be easily drawn by setting off the values of h and x calculated by these equations and given in the following table :—

θ in degrees	$\frac{x}{\frac{1}{2}l}=\frac{1}{\pi}(\sin\theta+\theta)$	$\frac{h}{h_0}=\cos\frac{1}{2}\theta$
0	0·000	1·000
20	0·220	0·985
40	0·427	0·940
60	0·609	0·866
80	0·758	0·766
100	0·869	0·643
120	0·942	0·500
140	0·982	0·342
160	0·998	0·174
180	1·000	0·000

As in the foregoing cases the curved outline of the column may be approximated by two pairs of straight lines touching the curve and sloping towards the ends of the column symmetrically about the cross section at the centre. Writing for shortness $\cdot h$ and $\cdot x$ for $h \div h_0$ and $x \div \frac{1}{2}l$, let the equation of a tangent be

$$\cdot h = c - m \cdot x, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (19)$$

touching the curve at the point $(\cdot x_c, \cdot h_c)$ so that

$$\cdot h_c = c - m \cdot x_c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (20)$$

and

$$\pi \cdot x_c = \sin\theta_c + \theta_c, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (21)$$

in which θ_c is given by

$$\cdot h_c = \cos\tfrac{1}{2}\theta_c.$$

The values of dx/dh found from (18) and (19) are equal at the point of contact and therefore

$$\pi\sqrt{1-\cdot h_c^2} = 4m \cdot h_c^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (22)$$

The volume of the column of approximately uniform strength is

$$V = \text{const.}(c^2 - cm + \tfrac{1}{3}m^2).$$

The condition of the least value of V leads to the equation

$$3 \cdot x_c(2c - m) = 3c - 2m. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (23)$$

It is not possible to determine the quantities $\cdot x_c$, $\cdot h_c$, c, and m by the direct solution of the last four equations. The following method of solution by approximation may however be employed. Let a curved outline of the column be drawn to scale by the help of the foregoing table. An inspection of this curve shows that when the volume V is least, $\cdot x$ will be not far from 0·4 or 0·5. Now calculate the numbers in the following table of the values of $\cdot V$ for three assumed values of θ, which roughly corresponds to $x = 0{\cdot}4$, 0·5, and 0·6.

Values of $\cdot V$

Angle θ assumed	$\cdot x = \frac{1}{\pi}(\sin\theta + \theta)$	$\cdot h = \cos\frac{1}{2}\theta$	$m = \frac{\pi}{2}\frac{\sin\frac{1}{2}\theta}{1+\cos\theta}$	$c = \cdot h + m \cdot x$	$\cdot V = c^2 - cm + \frac{1}{3}m^2$ $= (c - \frac{1}{2}m^2) + \frac{1}{12}m^2$
37°18′	·400114	·947490	·279770	1·059430	·852086
47°40′	·500122	·914725	·379293	1·104418	·848796
58°56′	·600061	·870642	·509686	1·176485	·871072

The abscissa of the point of contact for the minimum value of $\cdot V$ is then given by

$$\cdot x_c = \frac{(\cdot x_2^2 - \cdot x_3^2)\cdot V_1 + (\cdot x_3^2 - \cdot x_1^2)\cdot V_2 + (\cdot x_1^2 - \cdot x_2^2)\cdot V_3}{2\{(\cdot x_2 - \cdot x_3)\cdot V_1 + (\cdot x_3 - \cdot x_1)\cdot V_2 + (\cdot x_1 - \cdot x_2)\cdot V_3\}} = 0{\cdot}4630,$$

from which we obtain

$$\theta_c = 43^\circ\ 44', \qquad \cdot h_c = 0{\cdot}928, \qquad m = 0{\cdot}340,$$

$$c = 1{\cdot}085, \quad \text{and} \quad \cdot h_1 = c - m = 0{\cdot}746.$$

The geometrical construction of the approximate form of outline is as follows:—At the centre of the length set off a width equal to $1{\cdot}085\, h_0$ and at each end set off a width equal to $0{\cdot}746\, h_0$. Straight lines joining the ends of these widths form the tangents sought. These lines touch the curve (18) at distance $0{\cdot}463 \times \frac{1}{2}l$ from the centre, the width at that section being $0{\cdot}928\, h_0$.

The equations of condition of stability for columns of uniform strength already given by the formulæ (A_1), (A_2), and (A_3) may be adapted to the well-known semi-empirical expression for the strength of columns of uniform section, first proposed by Tredgold, revived by Lewis Gordon, and afterwards modified by Prof. Rankine, so as to be applicable to columns of any form of section. These formulæ of Rankine for columns with both ends rounded, with both ends flat, and with one end rounded and the other end flat, are respectively

$$\left.\begin{aligned} P &= \frac{fA}{1 + 4a\frac{l^2}{k^2}}, \\ P &= \frac{fA}{1 + a\frac{l^2}{k^2}}, \\ \text{and} \quad P &= \frac{fA}{1 + 2a\frac{l^2}{k^2}}. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ (R)$$

In these equations, P is the total breaking load on the column; f an experimental constant of strength for each kind of materials

in units of force per unit area; a another experimental constant, being an abstract number; A the sectional area of the column; l the length; and k the least radius of gyration of the section. It may be remarked here that the formulæ quoted by Prof. Unwin in his Machine Design as empirical rules suggested by Grashof are of the form

$$P = \frac{fAI}{\frac{Al^2}{C} + I}$$

and this is just the same as the first of Rankine formulæ (R); for, I/A is equal to k^2 and $4a$ may be written for the reciprocal of the constant C.

Values of the Constants f and a.

	Breaking f in lbs. per sq. in. excepting Grashof's values	$\frac{1}{a}$, (abstract number)	Authority
Wrought iron.....	36,000	36,000	Rankine.
Cast iron.....	80,000	6,400	
Dry timber.....	7,200	3,000	
Mild steel	42,000	36,000	Prof. Cotterill.
Wrought iron.....	10,000	22,400	Grashof. These f's are for greatest **safe** load.
Steel.....	12,000	20,000	
Cast iron.....	3,000	40,000	
	12,000	9,600	
Timber	900	6,000	
	500	10,400	
Mild steel	48,000	30,000	Prof. Fidler from experiments of Mr. Christie.
Hard steel	70,000	20,000	

Of the two values of P obtained from Grashof's constants, the lesser is to be adopted. For a very long and slender column, a "theoretical" value of the constant $4a$ in the first of the formulæ (R) is $f/\pi^2 E$, so that the ratio of the experimental to the theoretical value of the constant is $4\pi^2 Ea/f$. Now the following assumption is very likely to be approximately correct:—That for a column of uniform strength and of moderate length compared with the sectional dimensions, an empirical formula of the same form as Rankine's still holds good and that the same value of the ratio above mentioned likewise obtains in that formula. The result of this assumption is given in the following table:—

Columns with both Ends Rounded.

Form and kind of column		Theoretical formula	Semi-empirical formula	Theoretical value of coefficient $4a$
Of uniform section		$P=\pi^2\frac{EI}{l^2}$	$P=\frac{fA}{1+4a\frac{l^2}{k^2}}$	$\frac{f}{\pi^2 E}$
Of uniform strength	Width of section parallel to the plane of bending constant	$P=8\frac{EI}{l^2}$	$P=\frac{fA_0}{1+\frac{\pi^2 a}{2}\frac{l^2}{k_0^2}}$	$\frac{f}{8E}$
	Width of section perpendicular to the plane of bending constant	$P=\frac{64}{9}\frac{EI}{l^2}$	$P=\frac{fA_0}{1+\frac{9\pi^2 a}{16}\frac{l^2}{k_0^2}}$	$\frac{9f}{64E}$
	Sections similar figures	$P=\frac{3\pi^2}{4}\frac{EI}{l^2}$	$P=\frac{fA_0}{1+\frac{16a}{3}\frac{l^2}{k_0^2}}$	$\frac{4f}{3\pi^2 E}$

The last three of the empirical equations in the third column of the above table give the relation subsisting between the length of the column and the dimensions of the section at the centre of its length, for a given breaking load P. A_0 and k_0 are the area and radius of gyration of the said section.

For the finding of the relation between the dimensions of any section and the distance x of that section from the centre the following process may be employed. The equation (7) may be written thus:—

$$\frac{I}{I_0}=\frac{l^2-(2x)^2}{l^2}$$

Substituting this in (A_1) we have

$$P=8\frac{EI}{l^2-(2x)^2}.$$

Hence the first of the three empirical equations above referred to takes a more general form

$$P=\frac{fA}{1+\frac{\pi^2 a_1}{2}\frac{l^2-(2x)^2}{l^2}}, \quad \ldots\ldots\ldots\ldots\ldots \quad (B_1)$$

for a column of uniform strength, rounded at both ends, having a rectanglar section with the sides parallel to the plane of bending constant. See Fig. 5, Case I. This formula (B_1) applies equally well to columns of any form of section, provided that

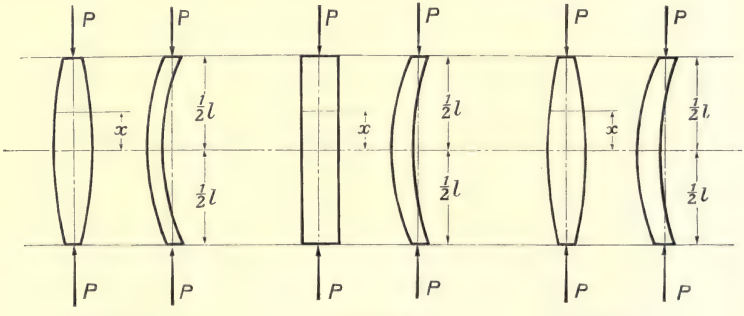


Fig. 5.

the dimensions parallel to the plane of bending remain constant. It may be noticed that at each end of the column, x become $\frac{1}{2}l$ so that the formula (B_1) reduces itself to the usual simple rule $P=fA$ for a short compression block.

The equation (13) may be written in the form

$$\frac{h^2}{h_0^2}\left(\frac{3}{4}+\frac{h}{4h_0}\right)=\frac{l^2-(2x)^2}{l^2}. \quad \ldots\ldots\ldots\ldots \quad (13')$$

The factor within the brackets on the left hand side of this equation is not much different from unity, as shown below.

$\frac{h}{h_0}$	1·0	0·8	0·6	0·4
$\frac{3}{4}+\frac{h}{4h_0}$	1·00	0·95	0·90	0·85

Since in Case II the width of a section perpendicular to the plane of bending is constant, we have $b=b_0$ so that

$$\frac{h^2}{h_0^2}=\frac{bh^2h_0}{b_0h_0^3}=\frac{k_0kA}{k_0^2A_0}.$$

Hence (13′) becomes

$$\frac{k_0kA}{k_0^2A_0}=\frac{l^2-(2x)^2}{l^2}, \text{ nearly,}$$

which being substituted in the equation (A_2) gives

$$P=\frac{64}{9}\frac{Ek_0kA}{l^2-(2x)^2}.$$

The second of the three empirical equations above referred to therefore takes a more general form

$$P=\frac{fA}{1+\frac{9\pi^2a}{16}\frac{l^2-(2x)^2}{k_0k}}, \quad \ldots\ldots\ldots\ldots \quad (B_2)$$

for a column of uniform strength, rounded at both ends, the dimensions of the section perpendicular to the plane of bending being constant. See Fig. 5, Case II. As in the previous case the formula (B_2) reduces itself at each end of the column to the usual simple rule $P=fA$ for a short compression block.

Now take equation (18). Expand each term on the right hand side in a series, square the resulting equation, and rearrange the terms. We have then

$$n^3\phi(n)=\frac{l^2-(2x)^2}{l^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (18')$$

in which

$$\phi(n)=\frac{4}{\pi}\left\{\left(\frac{2}{3}+\frac{1}{5}n^2+\frac{3}{28}n^4+\frac{5}{72}n^6+\frac{35}{704}n^8+\ldots\ldots\ldots\right)\right.$$
$$\left.-\frac{1}{\pi}\left(\frac{4}{9}n^3+\frac{4}{15}n^5+\frac{12}{175}n^7+\ldots\ldots\ldots\right)\right\}$$

and n stands for h/h_0. The function $\phi(n)$ is not much different from unity, as shown below.

$n=\frac{h}{h_0}$	1·0	0·8	0·6	0·4
$\phi(n)$	1·000	0·954	0·913	0·873

Since $$n^3=\frac{h^3}{h_0^3}=\frac{h_0h^3}{h_0^4}=\frac{k_0kA}{k_0^2A_0},$$

the equation (18′) becomes

$$\frac{k_0kA}{k_0^2A_0}=\frac{l^2-(2x)^2}{l^2}, \text{ nearly.}$$

Substituting this in (A_3) we obtain

$$P=\frac{3\pi^2}{4}\frac{Ek_0kA}{l^2-(2x)^2}.$$

The third of the three empirical equations above referred to therefore takes a more general form

$$P=\frac{fA}{1+\frac{16a}{3}\frac{l^2-(2x)^2}{k_0k}}, \quad \ldots\ldots\ldots\ldots\ldots \quad (B_3)$$

which likewise reduces itself at each end of the column to the simple rule $P=fA$ for a short compression block. The above formula (B_3) is for a column of uniform strength, rounded at both ends, having cross sections of similar figures. See Fig. 5, Case III.

In the application of the formula (B_1), (B_2), or (B_3) to the design of a column, the dimensions of the section at each end can be found by the simple rule $P=fA$; making x in the formula equal to zero, the dimensions of the section at the centre can be found in the same way as by Rankine's formula; and for other sections a few values of k lying between k_0 and the k at each end may be assumed, and the corresponding positions of the sections can be found by the solution of the formula for x, thus

$$\frac{x}{\frac{1}{2}l}=\sqrt{1-\frac{2k^2}{\pi^2al^2}\left(\frac{fA}{P}-1\right)}, \quad \ldots\ldots\ldots\ldots\ldots \quad (B_1')$$

$$\frac{x}{\frac{1}{2}l}=\sqrt{1-\frac{16k_0k}{9\pi^2al^2}\left(\frac{fA}{P}-1\right)}, \quad \ldots\ldots\ldots\ldots\ldots \quad (B_2')$$

or

$$\frac{x}{\frac{1}{2}l}=\sqrt{1-\frac{3k_0k}{16al^2}\left(\frac{fA}{P}-1\right)}. \quad \ldots\ldots\ldots\ldots\ldots \quad (B_3')$$

In a hollow cylindrical column under the action of an axial load there occurs secondary flexure tending to cause the buckling or wrinkling of the cylindrical shell and when the ratio of the diameter to the thickness is large, this secondary flexure becomes more predominant than the primary flexure of the column as a whole and failure may take place under a load very much less

than that indicated by Gordon-Rankine formula. Prof. W. E. Lilly of Trinity College, Dublin, carried out a series of experiments on a large number of small steel columns with rounded ends and from the results drew very important conclusions bearing on the point above mentioned. (Proceedings of the Inst. Mech. Eng., London, June 1905.) The columns tested were mild steel tubes ranging from $\frac{3}{8}''$ to $1''$ in external diameter and similar to those used in cycle construction.

Prof. Lilly's experiments show that the value of f in Gordon-Rankine formula depends on the failure of the tube by secondary flexure and that this secondary flexure does not depend on the length of the column, but only on the thickness and the radius of gyration, the relation of f to these quantities being given by the equation

$$f=\frac{F}{1+a_1\left(\frac{k}{t}\right)^2}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (L_1)$$

where F=the compressive strength of the material,

a_1=an experimental constant

$=\frac{1}{60}$ for mild steel,

t=the thickness of the tube in the same units as the radius of gyration k.

When $\frac{k}{t}$ approaches the limiting value of 0·5 for round solid bar, f is sensibly equal to F. For wrought iron the compressive strength F may be assumed at 24 or 25 tons per sq. inch and in Hodgkinson's experiments from which Rankine derived the constants in his formula, the average ratio d/t was approximately 16, that is, k/t=5·5 to 6·0. Substituting these values in the above formula and using $a_1=\frac{1}{60}$, f becomes 34000 to 37000 lbs. per sq. inch, which agrees very well with Rankine's value 36000 lbs.

per sq. inch. Hence the same value of a_1 may be employed both for mild steel and for wrought iron.

It is also shown by Prof. Lilly's experiments that for hollow circular columns there exists the most economical values of the ratios k/t and l/k expressed by the following relation :—

$$\left(\frac{l}{k}\right)^2 = 1000\left(\frac{k}{t} - \frac{1}{2}\right), \quad \ldots\ldots\ldots\ldots\ldots \quad (L_2)$$

in which the fraction $\frac{1}{2}$ on the right hand side may be neglected in comparison with large values of $\frac{k}{t}$. Then

$$k^3 = \frac{l^2 t}{1000}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (L_3)$$

For sections other than hollow circle, the above may be written in the form

$$k^2 = \frac{l\sqrt{A}}{94{\cdot}3}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (L_4)$$

which however requires to be verified by experiments.

Values of f by Prof. Lilly's Formula.

$\left(\frac{k}{t}\right)^2$	$\frac{d}{t}$	Value of $f=\frac{F}{1+\frac{1}{60}\left(\frac{k}{t}\right)^2}$ for F equal to		
		72000 lbs. per□″	63000 lbs. per□″	54000 lbs. per□″
120	31·0	24,000	21,000	18,000
84	25·9	30,000	26,300	22,500
60	21·9	36,000	31,500	27,000
42·9	18·5	42,000	36,800	31,500
30	**15·5**	**48,000**	**42,000**	**36,000**
20	12·6	54,000	47,300	40,500
12	9·75	60,000	52,500	45,000
5·45	6·53	66,000	57,800	49,500
5	6·24	66,500	58,200	49,800
4	5·57	67,500	59,100	50,600
3	4·80	68,600	60,000	51,400
2	3·87	69,700	61,000	52,300
1	2·65	70,800	62,000	53,100

The compressive strength of steel, F, for structural purposes may be taken at 32 to 28 tons or 72000 to 63000 lbs. per sq. inch and that of wrought iron at about 24 tons or 54000 lbs. per sq. inch. The accompanying table and diagram give the values of f calculated by Prof. Lilly's equation (L_1), corresponding to these assumed values of F. Granting the appropriateness of using these f's for actual struts and columns, it certainly demands urgent attention that a designer should caution himself against adopting too large a value of the ratio k/t, or against using too high a value of f for a large value of k/t. This remark applies to the use of Gordon-Rankine formulæ or other formulæ for columns of uniform section, as well as to the use of the foregoing formulæ for columns of uniform strength.

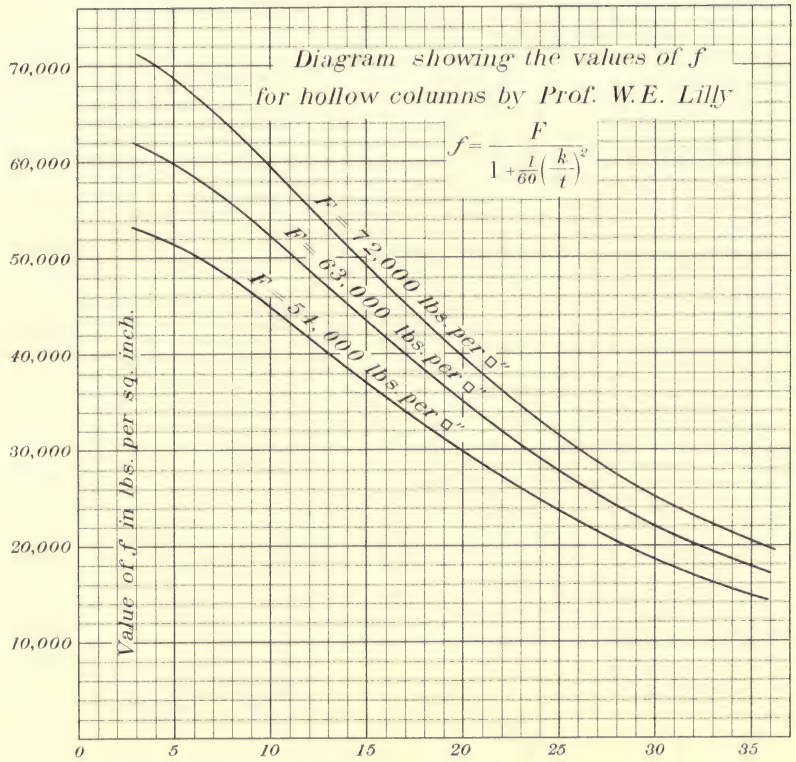


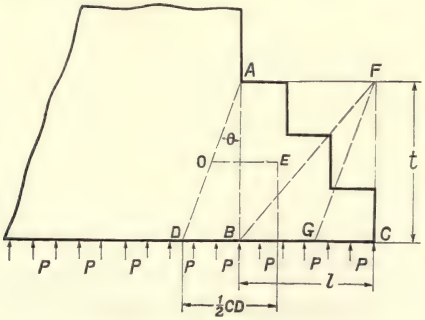
Diagram showing the values of f
for hollow columns by Prof. W.E. Lilly
$f = \frac{F}{1 + \frac{1}{60}\left(\frac{k}{t}\right)^2}$
F = 72,000 lbs. per □"
F = 63,000 lbs. per □"
F = 54,000 lbs. per □"
Value of f in lbs. per sq. inch.
70,000
60,000
50,000
40,000
30,000
20,000
10,000
0
5
10
15
20
25
30
35
Ratio of diameter to thickness, $\frac{d}{t}$.

XXVIII.

Strength of Foundation Footing for a Heavily Loaded Structure.

[*The Journal of the Engineering Society, Japan, No.* 311, *November*, 1908.]

Foundation courses of walls or chimneys have their width increased by a series of steps, with the object of distributing the pressure over a greater area than that of any bed joint in the body of the wall or chimney. The projecting portions of foundation footings thus formed have to resist considerable bending action when the structure is of great height. In the figure let *CDA* represent an axial vertical section of a chimney foundation supporting a very large weight. It may be assumed that the vertical downward pressure exerted by the bottom surface of the



foundation upon the excavated and suitably prepared bed is uniform all over the surface and of intensity p lbs. per square inch. If the base of the foundation is large compared with the projecting portions BC, the latter may safely be regarded as a cantilever of length l, loaded uniformly with vertical upward reactions of the bed, of intensity p lbs. per square inch.

Considering a vertical layer CDA, whose thickness is unity, and taking the vertical cross section AB of the cantilever, the bending moment at the section due to the uniform load p is $\frac{1}{2}pl^2$ and the moment of resistance of the section to bending is equal to $\frac{1}{6}ft^2$. Hence we have

$$\tfrac{1}{2}pl^2=\tfrac{1}{6}ft^2,$$

from which we obtain

$$f=\frac{3pl^2}{t^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

and

$$t=l\sqrt{\frac{3p}{f}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Now the bending stress f at the top or bottom edge of an oblique section such as AOD is greater than the f given by (1) and it will be found that there exists some particular direction of AOD for which the stress f is maximum. The bending moment at the section AOD is equal to the load on CD multiplied by the perpendicular let fall from the centre O of the section to the line of action of the resultant load, that is,

$$\begin{aligned} M&=p\times CD\times OE \\ &=p(l+t\tan\theta)(\tfrac{1}{2}CD-\tfrac{1}{2}t\tan\theta) \\ &=\tfrac{1}{2}pl(l+t\tan\theta). \end{aligned}$$

The moment of resistance of the section AOD is $\frac{1}{6}f(t\sec\theta)^2$.

We have therefore

$$\tfrac{1}{2}pl(l+t\tan\theta)=\tfrac{1}{6}ft^2\sec^2\theta,$$

from which we have

$$f=\frac{3pl^2}{t^2}\times(\cos^2\theta+\frac{t}{l}\cos\theta\sin\theta)$$

$$=\frac{3pl^2}{t^2}\times\tfrac{1}{2}(1+\cos 2\theta+m\sin 2\theta), \quad \ldots\ldots\ldots \quad (3)$$

in which m stands for t/l. For the particular direction of the section AOD determined by $\tan\theta=m$, the above value of the bending stress f is just equal to that given by the previous equation (1). For all smaller values of θ, the above value of f is greater than that given by (1) and become maximum when

$$\tan 2\theta=m. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Substituting this value of θ in (3) we obtain the greatest bending stress

$$f=\frac{3pl^2}{t^2}\times\tfrac{1}{2}(1+\sqrt{1+m^2}). \quad \ldots\ldots\ldots\ldots\ldots \quad (5)$$

If, as is sometimes the case in practice, the length of the projecting portion l is equal to the depth t, then the section AOD of the greatest stress makes an angle $\theta=22\frac{1}{2}^\circ$ with the vertical section AB and the value of that greatest stress is

$$f=\frac{3pl^2}{t^2}\times\tfrac{1}{2}(1+\sqrt{2})=1{\cdot}207\times\frac{3pl^2}{t^2},$$

which is greater by 21 per cent. than that given by the usual formula (1). Again if the depth t is $1\frac{1}{4}$ times the length l, the greatest bending stress f is 1·30 times the f given by (1).

The equation (4) gives us the following very simple geometrical construction for finding the direction of the section AOD for which the stress is greatest. Completing the rectangle $ABCF$,

draw the diagonal FB. Bisect the angle BFC by the line FG. Then AD drawn parallel to FG gives the required direction. The accompanying table gives the ratio of the greatest bending stress occurring at the section AOD to the bending stress at the vertical section AB.

$m=\frac{t}{l}$	0·7	0·8	0·9	1·0	1·1	1·2	1·3	1·4	1·5
$\frac{1}{2}(1+\sqrt{1+m^2})$	1·11	1·14	1·17	1·21	1·24	1·28	1·32	1·36	1·40

Solving the equation (5) for t we obtain

$$t=l\sqrt{\frac{3p}{f}+\tfrac{1}{4}\left[\frac{3p}{f}\right]^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

Suppose for example that the safe pressure p upon the foundation bed is one ton per square foot and the safe tensile stress f of the material of the foundation footing is three tons per square foot, that is, 46·7 lbs. per square inch, then according to the formula (2) the depth t would be equal to the length l; a more accurate value obtained by the above formula (6) is $t=1{\cdot}118\,l$.

XXIX.

Strength of Curved Beams.

By

A. Inokuty, *Kogakuhakushi, Kogakushi*

and

K. G. Takemura, *Kogakushi.*

[*The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. XIII, No.* 22, *May,* 1910.]

When the radius of curvature of the central longitudinal axis of a beam is not large compared with the dimensions of cross section of the beam, it is sometimes not accurate enough to assume that the intensity of the bending stress varies as the distance from the central axis of a cross section perpendicular to the plane of bending. In a curved beam, the form of a cross section may be quite symmetrical about the central axis perpendicular to the plane of bending and the action of the load on the beam may be simple bending, unaccompanied by direct axial pull or thrust and unaccompanied also by shearing force. In such a case, the bending stress is still greater at the outer skin on the concave side than on the convex side and the neutral surface of the beam lies on the inner side of the gravity axis, that is, on that side of the gravity axis on which the centre of

curvature lies. Crane hooks, chain links, &c. are examples of curved beams in which the effect of curvature has to be taken into consideration in the calculation of strength.

It will be assumed that the central line forming the locus of the centre of gravity of radial section lies wholly in one plane before and after bending. Referring to Fig. 1, let

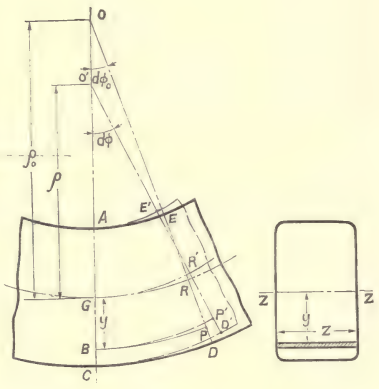


Fig. 1.

ρ_0 and ρ be the original and final radius of curvature of the central axis GR and GR' respectively, at the section ABC,

M the bending moment at the same section,

y the distance of points on radial cross section measured from the axis zz, passing through the $C.\ G.$ and perpendicular to the radius of curvature; this is reckoned positive when measured away from the centre of curvature and nagative when towards it,

p the bending stress of the fibre BP at distance y from the axis zz,

p_0 the bending stress of the fibre GR at $y=0$,

A the area of the cross section $=\int z dy$.

Taking another cross section $OERD$ at a very small distance $ds=GR$ measured on the gravity axis and supposing that the

first section ABC is kept in its original position, the second section will assume a position shown with accented letters *i.e.*, $E'R'D'$. We then have

original length $BP = (\rho_0 + y)d\phi_0$,

final length $BP' = (\rho + y)d\phi$.

Here it is assumed that y remains the same before and after bending. The bending stress of the fibre BP is given by

$$\frac{p}{E} = \frac{BP' - BP}{BP} = \frac{(\rho + y)d\phi}{(\rho_0 + y)d\phi_0} - 1.$$

Making y equal to zero, we obtain the bending stress of the fibre GR of the gravity axis,

$$\frac{p_0}{E} = \frac{\rho d\phi}{\rho_0 d\phi_0} - 1.$$

Eliminating $d\phi/d\phi_0$ we obtain

$$\frac{p}{E} = \left(1 + \frac{p_0}{E}\right)\frac{\rho_0}{\rho} \cdot \frac{\rho + y}{\rho_0 + y} - 1,$$

which may be written as follows:—

$$p = E\left\{\rho_0\left(1 + \frac{p_0}{E}\right)\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right)\frac{y}{\rho_0 + y} + \frac{p_0}{E}\right\}. \quad \ldots\ldots\ldots \quad (1)$$

Now p being the stress called into play by the bending moment alone, we must always have

$$\int p\, z\, dy = 0.$$

Substituting in this the value of p given by (1) and dividing all through by E, we have

$$\rho_0\left(1 + \frac{p_0}{E}\right)\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right)\int \frac{y\, z\, dy}{\rho_0 + y} + \frac{p_0}{E}\int z\, dy = 0,$$

$$\rho_0\left(1+\frac{p_0}{E}\right)\left(\frac{1}{\rho}-\frac{1}{\rho_0}\right)B+\frac{p_0}{E}A=0, \quad \ldots\ldots \quad (2)$$

where

$$B=\int\frac{y\,z\,dy}{\rho_0+y}. \quad \ldots\ldots \quad (3)$$

Again the bending moment at the section ABC must be equal to the moments of stresses on the section resisting bending action; we have therefore

$$M=\int y.\ p\,z\,dy$$

$$=E\rho_0\left(1+\frac{p_0}{E}\right)\left(\frac{1}{\rho}-\frac{1}{\rho_0}\right)\int\frac{y^2.\,z\,dy}{\rho_0+y}+p_0\int y.\,z\,dy$$

$$=E\rho_0\left(1+\frac{p_0}{E}\right)\left(\frac{1}{\rho}-\frac{1}{\rho_0}\right)C, \quad \ldots\ldots \quad (4)$$

where

$$C=\int\frac{y^2.\,z\,dy}{\rho_0+y}. \quad \ldots\ldots \quad (5)$$

Since y is measured from zz passing through the $C.\ G.$ of the section, the second integral in the above expression is zero. From (1) and (4) we have

$$p=\frac{M}{C}\frac{y}{\rho_0+y}+p_0. \quad \ldots\ldots \quad (6)$$

From (2) and (4) we have

$$\frac{M}{C}B+p_0A=0. \quad \ldots\ldots \quad (7)$$

Eliminating p_0 between the last two equations, we have

$$p=\frac{M}{C}\left(\frac{y}{\rho_0+y}-\frac{B}{A}\right). \quad \ldots\ldots \quad (8)$$

Now the integrals B and C can be reduced to simpler forms as follows:—

$$B=\int \frac{y.\,z\,dy}{\rho_0+y}=\int\left(1-\frac{\rho_0}{\rho_0+y}\right)z\,dy$$

$$=\int z\,dy-\rho_0\int\frac{z\,dy}{\rho_0+y}=A-A', \quad \ldots\ldots\ldots \quad (9)$$

where

$$A'=\rho_0\int\frac{z\,dy}{\rho_0+y}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

Again

$$C=\int\frac{y^2.\,z\,dy}{\rho_0+y}=\int\left(y-\rho_0+\frac{\rho_0^2}{\rho_0+y}\right)z\,dy$$

$$=\int y.\,z\,dy-\rho_0\int z\,dy+\rho_0^2\int\frac{z\,dy}{\rho_0+y}$$

$$=0-\rho_0 A+\rho_0 A'$$

$$=\rho_0(A'-A). \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

Substituting these values of the integrals in (8) we obtain

$$p=\frac{M}{\rho_0}\left\{\frac{y}{(A'-A)(y+\rho_0)}+\frac{1}{A}\right\}, \quad \ldots\ldots\ldots \quad (12)$$

or

$$p=\frac{M}{A(y+\rho_0)}\left\{1+\frac{A'}{A'-A}\frac{y}{\rho_0}\right\}, \quad \ldots\ldots\ldots \quad (12')$$

or

$$p=\frac{M}{\rho_0(A'-A)}\left\{\frac{A'}{A}-\frac{\rho_0}{y+\rho_0}\right\}. \quad \ldots\ldots\ldots\ldots \quad (12'')$$

At the neutral surface the bending stress p is zero and the position of this surface is found from (12'), thus

$$y_n=-\left(1-\frac{A}{A'}\right)\rho_0. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

The adoption of the function A', called a "modified area," is due to Prof. Arthur Morley, Professor of Mech. Eng. in the University College, Nottingham; the resulting equations (12) are given by him in this book on Strength of Materials, page 325, Edition 1908.

The value of the modified area A' can be calculated for simple geometrical forms. Take the case of a **rectangle.**

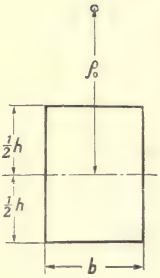


FIG. 2.

Here the width z is constant and equal to b, so that (10) becomes

$$A' = \rho_0 b \int_{-\frac{1}{2}h}^{\frac{1}{2}h} \frac{dy}{\rho_0 + y} = \rho_0 b \log (\rho_0 + y)_{-\frac{1}{2}h}^{\frac{1}{2}h}$$

$$= \rho_0 b \log \frac{2\rho_0 + h}{2\rho_0 - h}$$

$$= A n \log \frac{2n+1}{2n-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

where A is the area bh and n the ratio $\frac{\rho_0}{h}$.

The distance of the neutral surface from the centre of gravity of the section is by (13)

$$y_n = -\left(n - \frac{1}{\log \frac{2n+1}{2n-1}}\right) h. \quad \ldots\ldots\ldots\ldots\ldots \quad (15)$$

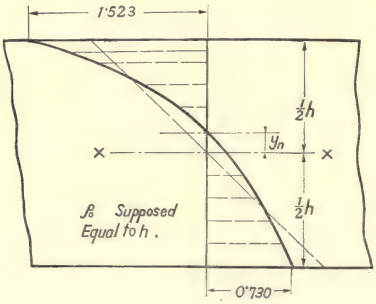


FIG. 3.

The bending stress p at a point distant y from the gravity axis of the section is

$$p=\frac{M}{\frac{1}{6}bh^2}\times\frac{n\log\frac{2n+1}{2n-1}-\frac{n}{\cdot y+n}}{6n\left(n\log\frac{2n+1}{2n-1}-1\right)}, \quad \ldots\ldots \quad (16)$$

where $\cdot y=y/h$, whose value ranges from $+\frac{1}{2}$ to $-\frac{1}{2}$.

If $\rho_0=h$, then $n=1$, $\log\frac{2n+1}{2n-1}=\log 3=1\cdot0986$,

$$y_n=-\left(1-\frac{1}{1\cdot0986}\right)h=-0\cdot0898\,h,$$

and

$$p=\frac{M}{\frac{1}{6}bh^2}\times 1\cdot690\left(1\cdot0986-\frac{1}{\cdot y+1}\right).$$

For a straight beam, this would become

$$p=\frac{M}{\frac{1}{6}bh^2}\times 2\cdot y.$$

In Fig. 3, the horizontal dotted ordinates of the curve give the bending stress p, the centre of curvature being situated toward the top of the page, and the value of p is

$$p=(\text{coef. in the last column of the table})\times\frac{M}{\frac{1}{6}bh^2}.$$

The following table gives the values of p at varying distance y for a curved beam of rectangular section with the radius of curvature ρ_0 equal to the depth of the beam h. For comparison the stress p for a straight beam of the same sectional dimensions is also given.

[TABLE.

y in fraction of depth h	Value of the coef. $2y$ for straight beam	Value of the coef. $1{\cdot}690\left(1{\cdot}0986-\frac{1}{y+1}\right)$ for curved beam
$-0{\cdot}5$	$-1{\cdot}0$	$-1{\cdot}523$
$-0{\cdot}45$	$-0{\cdot}9$	$-1{\cdot}216$
$-0{\cdot}4$	$-0{\cdot}8$	$-0{\cdot}960$
$-0{\cdot}3$	$-0{\cdot}6$	$-0{\cdot}558$
$-0{\cdot}2$	$-0{\cdot}4$	$-0{\cdot}256$
$-0{\cdot}1$	$-0{\cdot}2$	$-0{\cdot}021$
$y_n=-0{\cdot}0898$	$-0{\cdot}180$	0
0	0	$+0{\cdot}167$
$+0{\cdot}1$	$+0{\cdot}2$	$+0{\cdot}320$
$+0{\cdot}2$	$+0{\cdot}4$	$+0{\cdot}448$
$+0{\cdot}3$	$+0{\cdot}6$	$+0{\cdot}557$
$+0{\cdot}4$	$+0{\cdot}8$	$+0{\cdot}649$
$+0{\cdot}5$	$+1{\cdot}0$	$+0{\cdot}730$

Circular Section.

Here $y=r\sin\theta, \qquad z=2r\cos\theta.$

$$dy=r\cos\theta\, d\theta,$$

$$A'=\rho_0\int\frac{z\,dy}{y+\rho_0}=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{z\,r\cos\theta\,r\cos\theta}{\rho_0+r\sin\theta}d\theta$$

$$=\rho_0\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(-r\sin\theta+\rho_0-\frac{\rho_0^2-r^2}{\rho_0+r\sin\theta}\right)d\theta$$

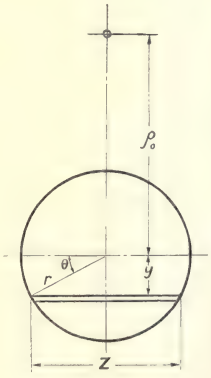


FIG. 4.

$$=2\rho_0\left[r\cos\theta+\rho_0\theta-(\rho_0^2-r^2)\frac{2}{\sqrt{\rho_0^2-r^2}}\tan^{-1}\left\{\sqrt{\frac{\rho_0-r}{\rho_0+r}}\tan\tfrac{1}{2}\left(\theta-\frac{\pi}{2}\right)\right\}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$=2\rho_0\left[0+\rho_0\frac{\pi}{2}-0-\left\{0+\rho_0\left(-\frac{\pi}{2}\right)-2\sqrt{\rho_0^2-r^2}\left(-\frac{\pi}{2}\right)\right\}\right]$$

$$=2\pi\rho_0(\rho_0-\sqrt{\rho_0^2-r^2})=8n(n-\sqrt{n^2-\tfrac{1}{4}})A, \quad \ldots\ldots\ldots\ldots \quad (17)$$

where n is the ratio ρ_0/dia. When the radius of curvature ρ_0 is equal to the diameter of the section, we have

$$A'=1{\cdot}071\ A.$$

For **Sections of any Form,** the function A' may be expanded into a series, of which first few terms may be used as an approximation to the value of A'. Thus

$$A'=\int\frac{\rho_0 z\,dy}{\rho_0+y}=\int\left(1-\frac{y}{\rho_0}+\frac{y^2}{\rho_0^2}-\frac{y^3}{\rho_0^3}+\frac{y^4}{\rho_0^4}-\ldots\ldots\ldots\right)z\,dy$$

$$=A+\frac{I}{\rho_0^2}-\frac{1}{\rho_0^3}\int y^3 z\,dy+\frac{1}{\rho_0^4}\int y^4 z\,dy \quad \ldots\ldots\ldots\ldots \quad (18)$$

where I is the moment of inertia of the section about the axis from which y is measured. The first two terms of the series gives an approximation to the value of A'. If the section is symmetrical about the ZZ axis, all the odd functions in the series disappear.

GRAPHIC METHOD OF FINDING MODIFIED AREA A'.

Let $APBKQF$ be any plane figure, of which ZCZ is the axis from which y is to be measured, C being the C. G. of the figure. Let O be the centre of curvature of the longitudinal axis of the beam. Draw a line AB parallel to ZCZ at a distance y

from it, meeting the perimeter of the figure at A and B. Join OA and OB, cutting the axis ZCZ at D and E.

Project these points on to the line AB, giving us the points A' and B', which are points on the perimeter of the required modified figure. The area of this figure is given by

$$A' = \int z' dy = \int \frac{\rho_0}{\rho_0 + y} z dy.$$

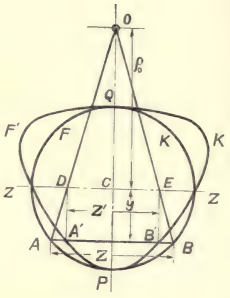


Fig. 5.

It may be noted that the point O from which OA and OB are drawn may be taken anywhere in the line drawn through the centre of curvature parallel to the axis ZCZ.

STRESSES IN HOOKS.

The formulæ (12) may be applied as a very good approximation for finding the stresses in the principal or horizontal section through the centre of curvature of a hook carrying a vertical load.

In the annexed figure let O be the centre of curvature of the points A and B of a horizontal section ADB, D being the centre of gravity of the section. Let a load W act at a distance l from D, so that $CD = l$ and $M = -Wl$. The uniformly distributed tensile stress on the section due to direct pull of the load W is W/A. In the formula (12″) make y equal to y_c and add the uniform stress W/A, and we obtain the resultant stress at A,

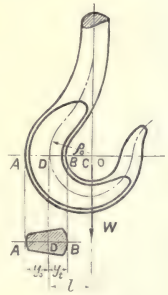


Fig. 6.

$$f_A = \frac{-Wl}{\rho_0(A'-A)}\left(\frac{A'}{A} - \frac{\rho_0}{\rho_0+y_c}\right) + \frac{W}{A}, \quad \ldots\ldots\ldots \quad (19)$$

which will turn out to be negative and consequently the stress is compressive on the convex side of the hook. In the same formula (12″) make y equal to $-y_t$ and add the uniform stress W/A, and we obtain the resultant stress at B,

$$f_B = \frac{-Wl}{\rho_0(A'-A)}\left(\frac{A'}{A} - \frac{\rho_0}{\rho_0-y_t}\right) + \frac{W}{A}, \quad \ldots\ldots\ldots \quad (20)$$

which will be positive showing that the stress is tensile on the concave side of the hook, as it of course should be.

GRAPHIC METHOD OF FINDING CENTROID, AND MOMENT OF INERTIA OF ANY PLANE FIGURE.

First Method.

To find centroid.

Referring to Fig. 7, let $APBKQF$ be any plane figure, the centroid of which is to be found. Take any reference axis XOX, and draw a line MN parallel to the reference axis at any convenient distance, as 10″, 15″, 20″ &c. Let AB be a line drawn parallel to the reference axis at a distance y from it, meeting the perimeter of the figure at A and B. Project these points on the line MN, giving us points D and E. Choose any point O in the axis XOX, preferably nearest to the figure, and join OD and OE, cutting the line AB and

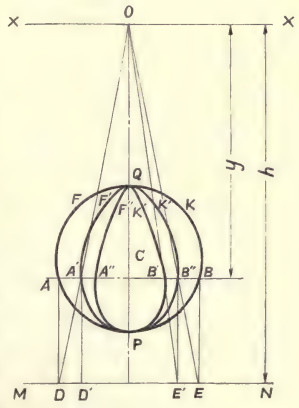


Fig. 7.

A' and B', which are points on the perimeter of the first modified figure. Finding a number of such points like A', and B', the first modified figure is easily drawn. Find the area of this new figure by means of a planimeter.

Let $AB=b$, and $A'B'=b'$.

Then the area of the original figure

$$A=\int b\,dy,$$

and the area of the first modified figure

$$A_1=\int b'\,dy=\int \frac{b}{h}.y.\,dy=\frac{1}{h}\int b\,y\,dy.$$

Therefore the distance of a line passing through the centroid of the original figure from the assumed reference axis XOX is

$$\bar{h}=\frac{\int b\,y\,dy}{\int b\,dy}=\frac{hA_1}{A}.$$

Taking any other reference axis and repeating a similar process, we obtain another line passing through the centroid of the original figure whose intersection with the line previously obtained is the centre of gravity of the original figure.

When the figure has an axis of symmetry PQ, the reference axis XOX is drawn perpendicular to the axis of symmetry, and the intersection of the axes gives us the pole O, and the line at the distance $\bar{h}$ from the reference axis XOX gives the centre of gravity, cutting the axis of symmetry at C.

To find moment of inertia.

Repeat a similar process on the first modified figure, projecting A', B' at D' and E', and getting A'' and B'' which are points of the perimeter of the second modified figure. Find the area of this new figure by means of a planimeter.

Let $A''B''=b''$.

Then the area of the second modified figure

$$A_2=\int b''\,dy$$

$$=\int\frac{b'}{h}y\,dy=\int\frac{\frac{b}{h}y}{h}y\,dy$$

$$=\frac{1}{h^2}\int b\,y^2\,dy=\frac{I}{h^2}.$$

Therefore the moment of inertia of the original figure about the reference axis XOX

$$I=h^2A_2.$$

Accordingly the moment of inertia of the figure about a parallel axis through its centre of gravity

$$I_0=I-(\bar{h})^2A$$

$$=h^2A_2-\left(\frac{hA_1}{A}\right)^2A$$

$$=h^2A_2-\frac{h^2A_1^2}{A}$$

$$=h^2\left(A_2-\frac{A_1^2}{A}\right).$$

Second Method.

Instead of using one fixed pole O, as we have just done, separate pole may be used for each line, like AB, the pole being the foot of the perpendicular from one of the extremities of the line AB on the reference axis.

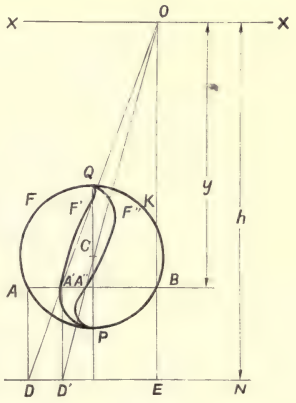


FIG. 8.

It is hardly necessary to note that the chosen extremities of the lines, such as AB, should always be either on the right or left hand side, and then the primeter of the derived figure on that side remains the same as that of the original figure, as shown in Fig. 8.

STRENGTH OF DRAW HOOKS USED BY I. G. R. OF JAPAN.

Fig. 9 shows one of these draw hooks. Taking the section of the hook through AB, let it be required to find the position of the centroid of the figure.

As the figure has an axis of symmetry OO' the reference axis XOX is drawn perpendicular to the axis of symmetry at the distance $\frac{7}{8}''$ from B (see Fig. 10), and the first modified figure is drawn by the second of the graphic methods of finding centroid.

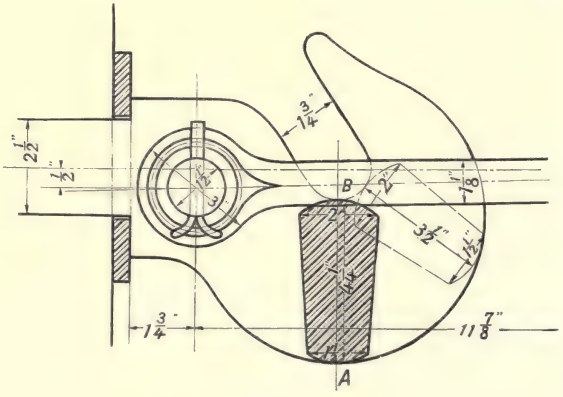


Fig. 9.

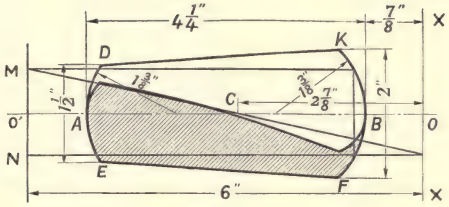


FIG. 10.

Here $h=6''$,

$A=7{\cdot}20$ sq. inches (by means of a planimeter),

$A_1=3{\cdot}45$ sq. inches (by means of a planimeter).

Therefore $$\bar{h}=\frac{hA_1}{A}=\frac{6\times 3{\cdot}45}{7{\cdot}20}=2{\cdot}88'',\quad \text{or}\quad 2\tfrac{7}{8}''\ \text{say}.$$

The second modified figure is then drawn on the first modified figure by using the second of the graphic methods of finding moment of inertia, and is shown by a hatched portion in Fig. 11.

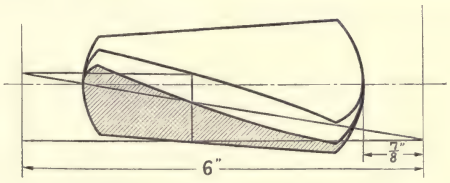


FIG. 11.

Here $A_2=1{\cdot}90$ sq. inches (by means of a planimeter).

Therefore $$I_0=h^2\left(A_2-\frac{A_1^2}{A}\right),$$

$$=6^2\left(1{\cdot}9-\frac{3{\cdot}45^2}{7{\cdot}20}\right),$$

$$=6^2(1{\cdot}9-1{\cdot}65)=36\times{\cdot}25,$$

$$=9{\cdot}0\ \text{(inches)}^4.$$

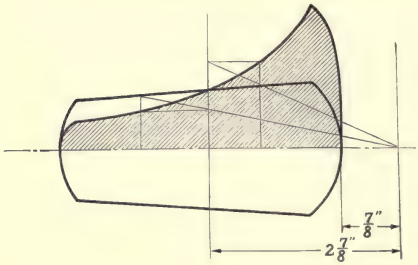


Fig. 12.

Fig. 12 shows the process of finding modified area A' of which the hatched portion is the required modified figure, the area being denoted by A'.

Here $A'=8{\cdot}70$ sq. inches (by means of a planimeter).

Now to find the stresses at A and B by using the equations (19) and (20).

The intensity of compressive stress at A

$$f_A=\frac{-Wl}{\rho_0(A'-A)}\left(\frac{A'}{A}-\frac{\rho_0}{\rho_0+y_c}\right)+\frac{W}{A}$$

$$=-W\left\{\frac{l}{\rho_0(A'-A)}\left(\frac{A'}{A}-\frac{\rho_0}{\rho_0+y_c}\right)-\frac{1}{A}\right\}.$$

Here $l=2\frac{7}{8}''$ (measured on Fig. 9),

$\rho_0=2\frac{7}{8}''$ (measured on Fig. 9),

$A'-A=1{\cdot}5$, $\quad\frac{A'}{A}=1{\cdot}21$,

$y_c=2\frac{1}{4}''$,

$\rho_0+y_c=5{\cdot}13$, $\quad\frac{1}{A}=\cdot139$.

Therefore $$f_A = -W\left\{\frac{1}{1\cdot5}\left(1\cdot21-\frac{2\cdot88}{5\cdot13}\right)-\cdot139\right\}$$

$$= -W\left\{\frac{1}{1\cdot5}(1\cdot21-\cdot56)-\cdot139\right\}$$

$$= -W(\cdot433-\cdot139)$$

$$= -\cdot294\,W, \text{ or } -\cdot3\,W, \text{ say.} \quad \ldots\ldots\ldots \text{(i)}$$

And the intensity of tensile stress at B

$$f_B = \frac{-Wl}{\rho_0(A'-A)}\left(\frac{A'}{A}-\frac{\rho_0}{\rho_0-y_t}\right)+\frac{W}{A}$$

$$= -W\left\{\frac{l}{\rho_0(A'-A)}\left(\frac{A'}{A}-\frac{\rho_0}{\rho_0-y_t}\right)-\frac{1}{A}\right\}.$$

Here $$y_t = 2'',$$

$$\rho_0 - y_t = \cdot88.$$

Therefore $$f_B = -W\left\{\frac{1}{1\cdot5}\left(1\cdot21-\frac{2\cdot88}{\cdot88}\right)-\cdot139\right\}$$

$$= -W\left\{\frac{1}{1\cdot5}(1\cdot21-3\cdot27)-\cdot139\right\}$$

$$= W(1\cdot373+\cdot139)$$

$$= 1\cdot512\,W, \text{ or } 1\cdot51\,W, \text{ say.} \quad \ldots\ldots\ldots \text{(ii)}$$

Therefore $$\frac{f_B}{f_A} = 5\cdot33. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

Next, let us find the values of f_A and f_B by using the formulæ given on page 535 of Vol. I of Prof. Unwin's Elements of Machine Design (1901 edition).

The formulæ are:—

$$f_A = \frac{-Wly_c}{I_0}+\frac{W}{A}$$

$$= -W\left\{\frac{ly_c}{I_0}-\frac{1}{A}\right\}.$$

$$f_B = \frac{W l y_t}{I_0} + \frac{W}{A}$$

$$= W\left\{\frac{l y_t}{I_0} + \frac{1}{A}\right\}.$$

Here $I_0 = 9{\cdot}0$ (inches)^4,

$l = 2\frac{7}{8}''$, $y_t = 2''$.

$y_c = 2\frac{1}{4}''$, $\frac{1}{A} = {\cdot}139$.

Therefore $$f_A = -W\left\{\frac{2{\cdot}88 \times 2{\cdot}25}{9{\cdot}00} - {\cdot}139\right\}$$

$$= -W({\cdot}72 - {\cdot}139)$$

$$= -{\cdot}581\,W, \quad \text{or} \quad -{\cdot}58\,W, \text{ say. .....} \quad \text{(iv)}$$

And

$$f_B = W\left\{\frac{2{\cdot}88 \times 2}{9{\cdot}00} + {\cdot}139\right\}$$

$$= W({\cdot}64 + {\cdot}139)$$

$$= {\cdot}779\,W, \quad \text{or} \quad {\cdot}78\,W, \text{ say. .....} \quad \text{(v)}$$

Therefore $$\frac{f_B}{f_A} = +1{\cdot}34, \quad \text{.....} \quad \text{(vi)}$$

which, compare with the previous result (iii). The estimated stresses calculated by the two methods are so widely different from each other that it is quite unsafe to neglect the effect of curvature in finding proper proportions of the draw hook.

XXX.

Method of Least Squares.

[*Hitherto unpublished.*]

Errors of Observation. The errors to which all observations are subject may be divided into systematic and accidental. Systematic errors are those which affect all observations on the same quantity in the same way and which are produced by well understood causes. As examples may be cited the errors arising from wrong graduation of the zero point of an instrument, those arising from the effect of temperature on the length of a measuring rod, those arising from the habit of the observer tending to make his readings err in a certain direction, &c.

Mistakes are errors committed by inexperienced and occasionally by the most skilled observers arising from temporary mental confusion. As such may be mentioned: mistakes of noting wrong figures such as 6 instead of 9; mistakes of sighting at wrong marks; mistakes in numerical calculations; mistakes that result from forgetting something that ought to have been done.

Accidental errors are those whose presence cannot be foreseen nor prevented; they are due to numerous causes. Such, for example, are the errors due to expansion of the instrument by solar heat or by a lamp, or due to the tilting of the instrument by

the weight of the observer, or due to the effect of the wind, or due to too much tightening of a screw, or due to inaccurate timing of observation, or due to inaccurate location of a mark, or due to inaccurate estimate of bisection, &c.

Before applying the Method of Least Squares, all systematic errors, and all evident mistakes should be carefully investigated and eliminated from the numerical results.

It may be noted here that some systematic errors may become accidental errors in certain cases of observations. Thus if a series of direct observations on the same quantity extend over a large number of days, taken at various hours of the day and night, the errors arising from the changes of atmospheric temperature may be considered as accidental errors, provided that the mean temperature for the whole period be the normal temperature for the series of observations.

The probability of an accidental error of a given magnitude is the ratio of the number of errors of that magnitude to the total number of errors. By the number of errors of a given magnitude is meant the number estimated per unit of the error of that magnitude. Thus if an error of magnitude x is chosen at random out of a very large number of errors and if the number of all the positive or negative errors that are numerically less than x is counted and found to be N, then we say that dN/dx is the number of errors of magnitude x.

Axioms derived from experience are :—

(1) Small errors are more frequent than large ones.

(2) Positive and negative errors are equally frequent.

(3) Very large errors do not occur.

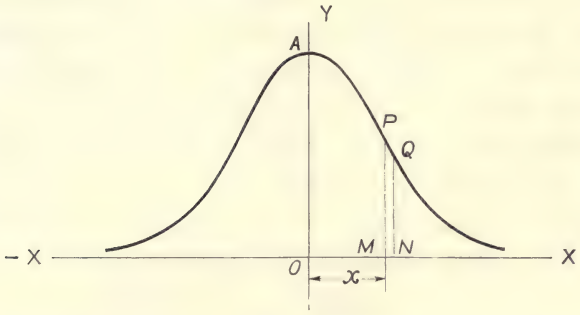


FIG. 1.

In a very large number of precise observations, then, the probability of a small error is greater than that of a large one; the probability of a positive error is the same as that of a negative error of the same magnitude; the probability of a very large error is zero. The words "very large" are somewhat vague, as it is impossible to assign a limiting magnitude to an error that will never occur. It is, therefore, best to extend the limit to infinity. In Fig. 1 taking rectangular axes OX and OY, let the abscissa x represent the magnitude of error, the ordinate y may then be considered to represent the probability corresponding to x, or it may be considered to represent the number of errors of magnitude x in a very large number of precise observations. The curve must then have the highest ordinates at the origin O [Axiom 1.]. The curve must be symmetrical about the axis OY [Axiom 2.]. The curve must be asymptotic to the axis OX at positive and negative infinity.

The probability of an error x must therefore depend on x^2 and must be expressed by some function whose value diminishes very rapidly as x increases. The probability that the error lies

between x and $x+\delta x$, where δx is very small, must be proportioned to δx. This is evident from Fig. 1, because the said probability is the number of errors lying between x and $x+\delta x$ divided by the whole number of errors in the observation, and that number is represented by the area of the elementary strip $MPQN$, of which the base MN is equal to δx.

Hence we may assume that the probability of an error of magnitude x being included in the range from x to $x+\delta x$ is

$$\phi(x^2)\delta x$$

Now the error must be included between $+\infty$ and $-\infty$ [Axiom 3.]. Hence as a first condition,

$$\int_{-\infty}^{+\infty}\phi(x^2)dx=1. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

When an event may happen in different independent ways, the probability of its happening is the sum of the separate probabilities. The integral is therefore the probability of committing an error whose magnitude lies between $+\infty$ and $-\infty$. The form of the function ϕ can be determined by the consideration of a very simple particular case.

Let a small shot be let fall from a certain height with the object of hitting a mark on a horizontal table. (Lord Kelvin Nat. Phil.)

Let two horizontal lines be drawn on the table through the mark at right angles to each other, and take them as axes of x and y respectively. The probability of the shot falling at a distance between x and $x+\delta x$ from the axis of y is $\phi(x^2)\delta x$. Similarly the probability of its falling between y and $y+\delta y$ from the axis of x is $\phi(y^2)\delta y$. The probability of its falling on the elementary area $\delta x\delta y$, whose coordinates are x, y, is the proba-

bility of a compound event produced by the concurrence of the two independent events above stated and is therefore the product of the separate probabilities, namely, $\phi(x^2)\phi(y^2)\delta x\,\delta y$ or $\phi(x^2)\phi(y^2)a$, where a denotes the indefinitely small area about the point xy. Had we taken any other rectangular axes through the same origin, the same probability should be $\phi(x'^2)\phi(y'^2)a$, in which $x'^2+y'^2=x^2+y^2$.

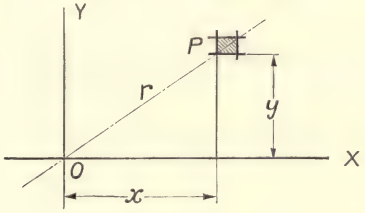


FIG. 2.

The probability of the error is thus independent of the direction of the error from the origin. Hence the same probability may be expressed by $\phi(r^2)a$, in which $r^2=x^2+y^2$. Hence we have

$$\phi(x^2)\phi(y^2)=\phi(x^2+y^2).$$

From this functional equation, we have

$$\phi(x^2)=A\,e^{mx^2},$$

where A and m are constants. Since the function has to diminish with the increase of x [Axiom 1.], m must be negative; and also since the exponent of e must be an abstract number, it is best to write $m=-\frac{1}{h^2}$, so that h represents the degree or measure of precision or coarseness of observation expressed in the same units as the observed quantity. Substituting in (1) we have

$$A\int_{-\infty}^{+\infty} e^{-\frac{x^2}{h^2}}dx=1,$$

whence $A=\frac{1}{h\sqrt{\pi}}$* and the probability of an error x lying between x and $x+\delta x$ is

$$\frac{1}{\sqrt{\pi}}e^{-\frac{x^2}{h^2}}\frac{\delta x}{h}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

This function is usually written in the form of equation to a curve, thus

$$y=\frac{1}{h\sqrt{\pi}}e^{-\frac{x^2}{h^2}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

This is called the **probability curve** and gives equal values of y for positive and negative values of x numerically equal. The maximum value of y corresponding to $x=0$, is equal to $1\div(h\sqrt{\pi})$. Hence the probability of the error 0 is $\frac{dx}{h\sqrt{\pi}}$. The differential coefficient of y is

$$\frac{dy}{dx}=-\frac{2}{h^3\sqrt{\pi}}e^{-\frac{x^2}{h^2}}x,$$

which becomes 0 when $x=0$ and also when $x=\pm\infty$, showing that the curve is horizontal over the origin and that the axis of x is an asymptote. The second differential coefficient is

$$\frac{d^2y}{dx^2}=-\frac{2}{h^3\sqrt{\pi}}\left(-\frac{2x^2}{h^2}+1\right)e^{-\frac{x^2}{h^2}},$$

which becomes 0 when $-\frac{2x^2}{h^2}+1=0$, showing that the curve has a point of inflection at $x=\pm\frac{1}{\sqrt{2}}h$.

Fig. 1 represents the probability curve. The quantity h is the smaller, the more precise the observations. The quantity

* See Williamson's Integ. Cal., page 151.

$\frac{1}{h\sqrt{\pi}}$ being proportional to the probability of error 0, the more precise the measurements, the larger is the value of $\frac{1}{h\sqrt{\pi}}$.

Probability Integral. The probability that an error x lies between any two given limits x_1 and x_2 is

$$P=\int_{x_1}^{x_2}\frac{1}{h\sqrt{\pi}}e^{-\frac{x^2}{h^2}}dx.$$

The probability that an error lies between the limits $-x$ and $+x$ is double the probability that it lies between 0 and $+x$.

Hence
$$P=2\int_0^x\frac{1}{h\sqrt{\pi}}e^{-\frac{x^2}{h^2}}dx \quad \ldots\ldots\ldots\ldots \quad (4_1)$$

expresses the probability that an error is numerically less than x. This may be written

$$P=\frac{2}{\sqrt{\pi}}\int_0^{\left(\frac{x}{h}\right)}e^{-\left(\frac{x}{h}\right)^2}d\left(\frac{x}{h}\right), \quad \ldots\ldots\ldots\ldots \quad (4_2)$$

which is called the probability integral. Writing $t=x/h$, the above becomes

$$P=\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt. \quad \ldots\ldots\ldots\ldots \quad (4_3)$$

Expanding e^{-t^2} by Maclaurin's theorem and effecting the integration, we have

$$P=\frac{2}{\sqrt{\pi}}\left(t-\frac{t^3}{3}+\frac{1}{1.2}\frac{t^5}{5}-\frac{1}{1.2.3}\frac{t^7}{7}+\ldots\ldots\right),$$

which is convenient for small values of t. For large values integrate by parts, thus

$$\int e^{-t^2} dx = -\frac{1}{2t} e^{-t^2} - \frac{1}{2} \int \frac{e^{-t^2}}{t^2} dt$$

$$= -\frac{1}{2t} e^{-t^2} + \frac{1}{2^2 t^3} e^{-t^2} + \frac{3}{2^2} \int \frac{e^{-t^2}}{t^4} dt.$$

And since $\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$, we have

$$\int_0^t e^{-t^2} dt = \frac{\sqrt{\pi}}{2} - \int_t^\infty e^{-t^2} dt,$$

from which we obtain

$$P = 1 - \frac{e^{-t^2}}{t\sqrt{\pi}} \left[1 - \frac{1}{2t^2} + \frac{1.3}{(2t^2)^2} - \frac{1.3.5}{(2t^2)^3} + \ldots\ldots\ldots \right].$$

From these two series the values of P can be found to any required degree of accuracy for all values of t, that is, $\frac{x}{h}$.

The Probable Error. In a series of errors the probable error has such a magnitude that an error taken at random is greater or less than the probable error is equally probable.

Let $r =$ the probable error of any one observation out of a large number of observations on a certain quantity. Then the above definition gives

$$\frac{2}{\sqrt{\pi}} \int_0^{\frac{r}{h}} e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \int_{\frac{r}{h}}^\infty e^{-t^2} dt = \frac{1}{2}.$$

The solution of this equation by trial and error, or by interpolation from a table of probability integral, gives

$$r = 0{\cdot}4769\, h. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

Propagation of Error. *Let X be a quantity measured directly many times with a probable error r. Required the probable error R of a linear function $mX+c$, in which m and c are known constants.*

In the function, all the errors of X appear multiplied m times and therefore

$$R=mr. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

Let X and Y be two independent quantities each measured directly many times with the resulting probable errors r_1 and r_2 respectively. Required to find the probable error R of $X+Y$ or of $X-Y$.

Let $$X \pm Y = Z,$$

and let x, y, and z be the errors of X, Y, and Z. The laws of errors x and y of the measured quantities X and Y are

$$\frac{1}{\sqrt{\pi}} e^{-\frac{x^2}{a^2}} \frac{dx}{a} \quad \text{and} \quad \frac{1}{\sqrt{\pi}} e^{-\frac{y^2}{b^2}} \frac{dy}{b}$$

respectively, a and b being the measures of precision for the two sets of observations. Now the probability of an error z lying between z and $z+dz$ is the same as the probability of concurrence of two independent events; (1) committing an error y within the range from y to $y+dy$, and (2) committiug an error x within the limits $+\infty$ and $-\infty$. But the range from y to $y+dy$ is the same as the range from $z-x$ to $z+dz-x$ (or from $z+x$ to $z+dz+x$). Hence the probability of error z lying between z and $z+dz$ is

$$\frac{1}{\pi a b} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{a^2}} dx \int_{z \mp x}^{z+dz \mp x} e^{-\frac{y^2}{b^2}} dy$$

$$= \frac{dz}{\pi a b} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{a^2}} e^{-\frac{(z \mp x)^2}{b^2}} dx$$

$$= \frac{dz}{\pi a b} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{a^2} - \frac{(z \mp x)^2}{b^2}} dx.$$

But $$\frac{x^2}{a^2} + \frac{(z \mp x)^2}{b^2} = \frac{a^2 + b^2}{a^2 b^2} \left(x \mp \frac{a^2 z}{a^2 + b^2} \right)^2 + \frac{z^2}{a^2 + b^2}.$$

Let $$u = \left(x \mp \frac{a^2 z}{a^2 + b^2} \right) \frac{\sqrt{a^2 + b^2}}{ab},$$

then $$dx = \frac{ab\, du}{\sqrt{a^2 + b^2}},$$ and also when

$$x = \pm\infty, \quad u = \pm\infty.$$

The above integral therefore becomes

$$\frac{dz}{\pi ab} \int_{-\infty}^{+\infty} e^{-u^2} e^{-\frac{z^2}{a^2 + b^2}} \frac{ab}{\sqrt{a^2 + b^2}} du$$

$$= \frac{dz}{\pi \sqrt{a^2 + b^2}} e^{-\frac{z^2}{a^2 + b^2}} \int_{-\infty}^{+\infty} e^{-u^2} du$$

$$= \quad \text{,,} \quad \times \sqrt{\pi}$$

$$= \frac{1}{\sqrt{\pi}} e^{-\frac{z^2}{a^2 + b^2}} \frac{dz}{\sqrt{a^2 + b^2}}.$$ (Lord Kelvin Nat. Phil.)

The measure of precision h of the observations Z has, therefore, the value $\sqrt{a^2 + b^2}$. Hence the required probable error of $X + Y$ or $X - Y$ is, by (5),

$$R = \cdot 4769 \sqrt{a^2 + b^2}$$

$$= \cdot 4769 \sqrt{\left(\frac{r_1}{\cdot 4769} \right)^2 + \left(\frac{r_2}{\cdot 4769} \right)^2}$$

$$= \sqrt{r_1^2 + r_2^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

The probable error R of a linear function

$$m_0+m_1X_1+m_2X_2+\ldots\ldots+m_nX_n$$

of independent quantities X_1, X_2,.....X_n, which are measured directly many times, with the resulting probable errors r_1, r_2,.....r_n, is given by

$$R=\sqrt{(m_1r_1)^2+(m_2r_2)^2+\ldots\ldots+(m_nr_n)^2}, \quad \ldots\ldots \quad (8)$$

m_0, m_1, m_2,.....m_n being known constants.

Let $Z=\phi(X_1, X_2,\ldots\ldots X_n)$ be any function of independent quantities X_1, X_2,.....X_n, each observed directly many times and let the errors x_1, x_2,..... x_n be supposed small compared with the true values a_1, a_2,.....a_n. Expanding ϕ and neglecting powers and products of small quantities x_1, x_2,.....x_n, we obtain

$$\begin{aligned} Z&=\phi(X_1, X_2,\ldots\ldots X_n)\\ &=\phi(a_1+x_1, a_2+x_2,\ldots\ldots a_n+x_n)\\ &=\phi(a_1, a_2,\ldots\ldots a_n)\\ &\quad+x_1\frac{d\phi}{da_1}+x_2\frac{d\phi}{da_2}+\ldots\ldots+x_n\frac{d\phi}{da_n}. \end{aligned}$$

By the last case, the probable error R of the function is given by

$$R^2=r_1^2\left(\frac{d\phi}{dX_1}\right)^2+r_2^2\left(\frac{d\phi}{dX_2}\right)^2+\ldots\ldots+r_n^2\left(\frac{d\phi}{dX_n}\right)^2. \quad \ldots\ldots \quad (9)$$

Example. Let c be the probable error of the diameter D of a circle measured many times. Find the probable error of the circumference and of the area.

The probable error of the circumference is πc and that of the area is

$$R = r_1\left(\frac{d\phi}{dX_1}\right)$$

$$= c\frac{d}{dD}\left(\frac{\pi}{4}D^2\right)$$

$$= c \times \frac{\pi}{4} \times 2D = \frac{\pi}{2}cD.$$

Example. The measurements of two sides of a rectangle gave $a \pm r_1$, and $b \pm r_2$, a and b being the most probable values of the lengths of the sides, and r_1 and r_2 the corresponding probable errors. Required the probable error of the area ab.

$$R = \sqrt{r_1^2\left(\frac{d\phi}{dX_1}\right)^2 + r_2^2\left(\frac{d\phi}{dX_2}\right)^2}$$

$$= \sqrt{r_1^2\left[\frac{d}{da}(ab)\right]^2 + r_2^2\left[\frac{d}{db}(ab)\right]^2}$$

$$= \sqrt{b^2r_1^2 + a^2r_2^2},$$

and $$\frac{R}{ab} = \sqrt{\left(\frac{r_1}{a}\right)^2 + \left(\frac{r_2}{b}\right)^2}.$$

Example. With the data in the last example, find the probable error of the diagonal.

Here $$\phi = (a^2+b^2)^{\frac{1}{2}}, \qquad \frac{d\phi}{da} = \frac{a}{\sqrt{a^2+b^2}},$$

and $$\frac{d\phi}{db} = \frac{b}{\sqrt{a^2+b^2}}, \quad \text{so that} \quad R = \sqrt{\frac{(ar_1)^2 + (br_2)^2}{a^2+b^2}}.$$

Example. The probable error of a number a is r. Find the probable error of $\sqrt{a}$.

$$R = r\frac{d\phi}{da} = r\frac{d}{da}\left(a^{\frac{1}{2}}\right)$$

$$= \frac{r}{2\sqrt{a}}.$$

This probable error expressed as a fraction of the root $\sqrt{a}$ is equal to one half of r expressed as a fraction of the number a. Similarly the probable error of cube root is

$$R=\frac{r}{3\sqrt[3]{a^2}},$$

and

$$\frac{R}{\sqrt[3]{a}}=\frac{1}{\sqrt[3]{a}}\times\frac{r}{3\sqrt[3]{a^2}}=\frac{1}{3}\left(\frac{r}{a}\right).$$

Example. Find the probable error of a^n, that of a being known.

$$R=r\frac{d\phi}{da}=r\frac{d}{da}\left(a^n\right)$$

$$=r\,n\,a^{n-1},$$

and

$$\frac{R}{a^n}=\frac{r\,n\,a^{n-1}}{a^n}=n\left(\frac{r}{a}\right),$$

being n times the probable error of the number expressed as a fraction of the number itself.

Example. The probable error of the common logarithm of a number a is r. Find the probable error of the number itself.

Let $l=\log_{10} a$, so that

$$\phi(l)=a^{10}=10^l.$$

$$\frac{d\phi}{dl}=10^l\log_e 10=a\log_e 10=2{\cdot}303\,a,$$

and

$$R=r\frac{d\phi}{dl}$$

$$=2{\cdot}303ar.$$

Example. A process of multiplication and division of n numbers by one another is performed by means of a table of logarithms. Find the probable error of the result of the calculation.

The probable error of a logarithm taken from a table of logarithms is $\pm\frac{1}{4}$ unit of the last figure. The probable error of the resulting logarithm is

$$R=\sqrt{(\tfrac{1}{4})^2+(\tfrac{1}{4})^2+(\tfrac{1}{4})^2+\ldots\ldots \text{ to } n \text{ terms}}$$
$$=\tfrac{1}{4}\sqrt{n}.$$

Hence by the last example, the probable error of the number N corresponding to the resulting logarithm is

$$2{\cdot}303\times\tfrac{1}{4}\sqrt{n}\times N,$$

in units of the last figure of the logarithm. This probable error expressed as a fraction of the number itself is

$$\frac{2{\cdot}303\times\frac{1}{4}\sqrt{n}}{L},$$

where L is 1 followed by as many ciphers as there are places in the logarithm.

Example. Find the probable error of nth power of a number a calculated by means of a table of common logarithms.

Let $l=\log_{10} a.$

Then the nth power of the number is

$$N=a^n=10^{nl},$$
$$\frac{dN}{dl}=n10^{nl}\log_e 10$$
$$=2{\cdot}303\, n\, N.$$

Therefore the probable error of N is

$$R=\frac{1}{4L}\times 2{\cdot}303\, n\, N.$$

If, for example, $(1{\cdot}048)^{30}$ is calculated by means of a table of five place logarithms, we have

$$\log N = n \log a = 30 \times \log 1{\cdot}048$$

$$= 30 \times 0{\cdot}02036$$

$$= 0{\cdot}61080$$

$$= \log 4{\cdot}0813,$$

so that $N = 4{\cdot}0813.$

The probable error of N is

$$R = \frac{1}{4L} \times 2{\cdot}303\, n\, N$$

$$= \frac{2{\cdot}303 \times 30 \times 4{\cdot}0813}{4 \times 100000}$$

$$= 0{\cdot}0007048.$$

The result may be written thus

$$4{\cdot}0813 \pm 0{\cdot}0007.$$

The probable error of a^n calculated by logarithms varies directly as the exponent n and inversely as 10 raised to the power of the number of places in the logarithm and hence a high power of a number is liable to a considerable error if calculated by short logarithms.

Example. In the measurement of a discharge of water by a standard orifice, the formula employed is

$$Q = c \frac{\pi}{4} D^2 \sqrt{2gh},$$

in which Q is the discharge in cubic units per unit time, c the coefficient of discharge, D the diameter of the orifice, and h the head over the centre of the orifice. If r_1 and r_2 are the probable errors of D and h respectively, find the probable error of Q.

The probable error of Q is

$$R=\sqrt{r_1^2\left(\frac{dQ}{dD}\right)^2+r_2^2\left(\frac{dQ}{dh}\right)^2}$$

$$=c\frac{\pi}{4}\sqrt{2g}\sqrt{r_1^2(2Dh^{\frac{1}{2}})^2+r_2^2(\tfrac{1}{2}h^{-\frac{1}{2}}D^2)^2}$$

$$=c\frac{\pi}{4}D^2\sqrt{2gh}\,\sqrt{4\left(\frac{r_1}{D}\right)^2+\frac{1}{4}\left(\frac{r_2}{h}\right)^2},$$

Therefore $$\frac{R}{Q}=\sqrt{4\left(\frac{r_1}{D}\right)^2+\frac{1}{4}\left(\frac{r_2}{h}\right)^2}.$$

Example. The length of a line is equal to $a\pm r_1$ and the angle which the line makes with a reference axis is $\alpha\pm r_2$, r_1 and r_2 being the probable errors of a and α respectively. Find the probable error of $a\sin\alpha$.

It being assumed that r_2 is expressed in radians, the probable error of

$$\phi(a,\alpha)=a\sin\alpha$$

is

$$R=\sqrt{r_1^2\left(\frac{d\phi}{da}\right)^2+r_2^2\left(\frac{d\phi}{d\alpha}\right)^2}$$

$$=\sqrt{r_1^2\sin^2\alpha+r_2^2(a\cos\alpha)^2}$$

$$=a\sin\alpha\sqrt{\left(\frac{r_1}{a}\right)^2+\left(r_2\cot\alpha\right)^2}.$$

If for example $a=100$ ft., $r_1=0{\cdot}05$ ft., $\alpha=45^\circ$, and $r_2=1'=0{\cdot}0002908$ radian, we have

$$R=100\times0{\cdot}7071\sqrt{\left(\frac{{\cdot}05}{100}\right)^2+(0{\cdot}0002908\times1)^2}$$

$$=70{\cdot}71\times\frac{5{\cdot}784}{10000}=0{\cdot}0409,$$

and the result may be written in the form $70{\cdot}71\pm0{\cdot}04$ ft.

Principle of Least Squares. Let n equally good measurements be made upon functions of one or more unknown quantities. The corresponding errors being $x_1, x_2, x_3, \ldots\ldots x_n$, the probabilities of these errors being within a range dx are

$$\frac{dx}{h\sqrt{\pi}}\, e^{-\frac{x_1^2}{h^2}}, \quad \frac{dx}{h\sqrt{\pi}}\, e^{-\frac{x_2^2}{h^2}}, \quad \ldots\ldots\ldots \quad \frac{dx}{h\sqrt{\pi}}\, e^{-\frac{x_n^2}{h^2}}$$

respectively, h being the same in all, because all the observations are supposed to be equally good. Now the probability of the compound event of committing the system of independent errors $x_1, x_2, \ldots x_n$ is the product of these separate probabilities, that is,

$$P = \frac{(dx)^n}{h^n (\pi)^{\frac{n}{2}}}\, e^{-\frac{1}{h^2}\left(x_1^2 + x_2^2 + \ldots\ldots\ldots + x_n^2\right)}.$$

Different values given to the unknown quantities in the functions give rise to different values of $\Sigma(x^2)$ and consequently to different values of the probability P, and the most probable values of the unknown quantities will be those that make P a maximum. But P is maximum when $\Sigma(x^2)$, the exponent of e, is a minimum. Thus *the most probable values of the quantities to be determined are those which make the sum of the squares of errors the least possible,* that is

$$\Sigma\,(x^2) \text{ the least.}$$

This is the principle of least squares.

An accidental error x is the difference between the true value of the observed quantity and the result of measurement upon it. A residual is the difference between the most probable value of the observed quantity and the measurement upon it. Evidently the most probable value will approach more nearly to the true value, the greater the number of observations, as likewise the

residuals to the errors. With an infinite number of good observations, the most probable value will become the same as the true value and the residuals will coincide with the errors. With a large number of good observations, the residuals will differ but slightly from the errors, so that the principles and laws concerning the residuals will be essentially the same as for errors. For this reason, residuals are often spoken of as residual errors and sometimes simply as errors. We use the letter v to represent a residual.

The fact that $\Sigma(x^2)$ has to satisfy the condition of being the least possible implies that we are in reality considering all x's as residual errors and not as true errors. Hence the statement of the principle of Least Squares as applied in practice is

$$\Sigma v^2 \text{ should be the least possible,} \quad \ldots\ldots\ldots\ldots \quad (10)$$

in which $v_1, v_2, v_3, \ldots v_n$ are residuals corresponding to $x_1, x_2, x_3, \ldots x_n$.

Arithmetical Mean. Let $m_1, m_2, m_3, \ldots m_n$ be the direct measurements upon a quantity made with equal care, and Z the most probable value of the quantity which is to be determined. Then the residual errors are $(Z-m_1), (Z-m_2), \ldots\ldots(Z-m_n)$, and from the fundamental principle (10), we have to make

$$(Z-m_1)^2+(Z-m_2)^2+\ldots\ldots\ldots+(Z-m_n)^2$$

a minimum. Differentiating this expression with respect to Z and equating the result to zero, we obtain

$$Z=\frac{1}{n}(m_1+m_2+\ldots\ldots\ldots+m_n), \quad \ldots\ldots\ldots\ldots\ldots \quad (11)$$

that is, the most probable value of a directly measured quantity is the arithmetical mean of the observations.

Weights of Observations. The weight assigned to an observation expresses the number of observations of standard accuracy, of which it is an equivalent. Thus the arithmetical mean of p equally good direct measurements has a weight p, the weight of each single measurement being unity. And inversely any observation having a weight p may be regarded as the equivalent of p observations of standard accuracy, each of weight unity.

Let $p_1, p_2, \ldots\ldots p_n$ be the weights of direct observations $m_1, m_2, \ldots\ldots m_n$ made upon a quantity Z; the most probable value of the quantity is then evidently

$$Z = \frac{p_1 m_1 + p_2 m_2 + \ldots\ldots + p_n m_n}{p_1 + p_2 + \ldots\ldots + p_n}. \quad \ldots\ldots \quad (12)$$

This is sometimes called the general mean, or weighted mean.

Now let us consider the more general case of indirect measurements having different degrees of precision, or different weights. The problem may be stated as follows:—

Let n measurements be made upon a function of one or more unknown quantities; and let $p_1, p_2, \ldots\ldots p_n$ be the respective weights and $x_1, x_2, \ldots\ldots x_n$ the corresponding errors. Also let $h_1, h_2, \ldots\ldots h_n$ be the respective measures of precision.

Then the probablity of committing the system of independent errors $x_1, x_2, \ldots\ldots x_n$ is

$$P = \frac{(dx)^n}{h_1 h_2 \ldots\ldots h_n (\pi)^{\frac{n}{2}}} e^{-\left(\frac{x_1^2}{h_1^2} + \frac{x_2^2}{h_2^2} + \ldots\ldots + \frac{x_n^2}{h_n^2}\right)}.$$

The most probable values of the unknown quantities will be those that make P a maximum. Hence we must have

$$\frac{x_1^2}{h_1^2} + \frac{x_2^2}{h_2^2} + \ldots\ldots + \frac{x_n^2}{h_n^2} \text{ a minimum.}$$

Now let z_1, z_2, and z_3 be the unknown quantities and Z their function. Then the errors x_1, x_2,.....x_n are the values of $Z-m_1$, $Z-m_2$,.....$Z-m_n$, respectively, m_1, m_2,.....m_n being the calculated values of Z obtained from observed data, or the values of Z directly measured. The above condition of obtaining the most probable values of z_1, z_2, and z_3 then becomes

$$\frac{(Z-m_1)^2}{h_1^2}+\frac{(Z-m_2)^2}{h_2^2}+.....+\frac{(Z-m_n)^2}{h_n^2} \text{ a minimum.}$$

Differentiating and equating the result to zero, we have

$$Z=\frac{\frac{m_1}{h_1^2}+\frac{m_2}{h_2^2}+.....+\frac{m_n}{h_n^2}}{\frac{1}{h_1^2}+\frac{1}{h_2^2}+.....+\frac{1}{h_n^2}}. \quad ..... \quad (13)$$

If the h's in this equation are replaced by r's given by (5), the result becomes

$$Z=\frac{\frac{m_1}{r_1^2}+\frac{m_2}{r_2^2}+\frac{m_3}{r_3^2}+.....+\frac{m_n}{r_n^2}}{\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+.....+\frac{1}{r_n^2}}. \quad ..... \quad (13')$$

Comparing this with (12), we at once see that the weight is inversely as the square of the measure of precision, so that

$$p_1 h_1^2=p_2 h_2^2=p_3 h_3^2=.....=h^2, \quad ..... \quad (14)$$

in which h is a constant standard measure of precision, corresponding to weight unity. The above may be written thus

$$\frac{1}{h_1^2}=\frac{p_1}{h^2}, \quad \frac{1}{h_2^2}=\frac{p_2}{h^2}, \quad ....., \quad \frac{1}{h_n^2}=\frac{p_n}{h^2}. \quad ... \quad (14')$$

Hence the condition of obtaining the most probable values of z_1, z_2, and z_3 in the case of observation with different degrees of precision, or with different weights, is

$$\Sigma \frac{v^2}{h^2} \text{ a minimum, } \ldots\ldots\ldots\ldots (15)$$

or $$\Sigma p v^2 \text{ a minimum. } \ldots\ldots\ldots\ldots (16)$$

An observation equation is an equation expressing the relation between measurements and the quantities sought. Thus supposing that there exists a relation $I=aB+c$ between the indicated horse power I and the brake horse power B of an engine, with two constants a and c, let simultaneous measurements be made, giving I_1 and B_1 for the values of I and B. Then $I_1=aB_1+c$ is an observation equation. Two such equations can determine the values of a and c accurately, if the measurements be perfect and the measured relation rigorously true. Generally, however, such is not the case, and good many sets of observations are required for the proper adjustment of observations so as to obtain the most probable values of the constants a and c.

Formation of Normal Equations from Observations of Equal Weight.

Consider the case of observations of equal precision or of equal weight. Let there be, say, three unknown quantities z_1, z_2, z_3 and let the equation between them and the measured quantities be linear and of the form

$$Z=az_1+bz_2+cz_3,$$

in which Z is the measured quantity, and a, b, c are either known and quite determinate in values for each measurement of Z, or to be measured simultaneously with Z. The observation equations are then

$$Z_1=a_1 z_1+b_1 z_2+c_1 z_3,$$

$$Z_2=a_2 z_1+b_2 z_2+c_2 z_3,$$

$$Z_3 = a_3 z_1 + b_3 z_2 + c_3 z_3,$$

.....

$$Z_n = a_n z_1 + b_n z_2 + c_n z_3,$$

the suffix n representing the number of observation equations, arising from n sets of observations. The number of these equations, n, must be greater than the number of unknown functions, three in this case. Each one of the above equations is an approximation to the truth. Considering z_1, z_2, z_3 to be now the most probable values of these quantities, we have

$$a_1 z_1 + b_1 z_2 + c_1 z_3 - Z_1 = v_1,$$

$$a_2 z_1 + b_2 z_2 + c_2 z_3 - Z_2 = v_2,$$

$$a_3 z_1 + b_3 z_2 + c_3 z_3 - Z_3 = v_3,$$

.....

$$a_n z_1 + b_n z_2 + c_n z_3 - Z_n = v_n,$$

where v_1, v_2, v_3,.....v_n are small residuals. Now by the condition (10), we have

$$\frac{d}{dz_1} \Sigma v^2 = \Sigma 2v \frac{dv}{dz_1}$$

$$= \Sigma 2(az_1 + bz_2 + cz_3 - Z)a = 0.$$

Hence

$$\Sigma aZ = \Sigma a^2.z_1 + \Sigma ab.z_2 + \Sigma ac.z_3.$$

Differentiating Σv^2 with respect to z_2 and equating the result to zero, we obtain

$$\Sigma bZ = \Sigma ab.z_1 + \Sigma b^2.z_2 + \Sigma bc.z_3,$$

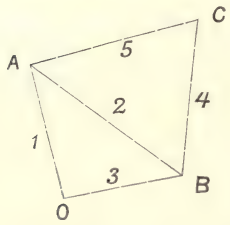
similarly

$$\Sigma cZ = \Sigma ac.z_1 + \Sigma bc.z_2 + \Sigma c^2.z_3. \quad \text{..... (17)}$$

These equations are called normal equations, their number being always equal to the number of unknown quantities. The solution of these equations gives the values of the unknown quantities.

The rule for forming the normal equations is as follows :— Mutiply all the observation equations by the respective coefficient of one of the unknown quantities, say, z_1. Add together all the resulting equations and we obtain one normal equation. Do the same operations for the other unknown quantities z_2 and z_3, to obtain two more equations. There will in general be as many normal equations as there are unknown quantities.

Example. The elevations of three points A, B, C above a given bench mark O are to be determined. Five lines of levels are run between these points, giving the following results :—



A above $O=10$ ft.

B above $A=7$ ft.

B above $O=18$ ft.

B above $C=9$ ft.

C below $A=2$ ft.

Let the elevations of the points A, B, and C be designated by z_1, z_2, and z_3, then the following observation equations may be written :—

$$z_1=10,$$

$$z_2-z_1=7,$$

$$z_2=18,$$

$$z_2-z_3=9,$$

$$z_1-z_3=2.$$

The coefficients of z_1 in these equations are 1, -1, 0, 0, 1 respectively. These being multiplied into the respective equations, and the results added together, we obtain one normal equation

$$3z_1-z_2-z_3=5. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

The coefficients of z_2 are 0, 1, 1, 1, 0 respectively. These being multiplied into the respective equations, and the results added together, we obtain

$$-z_1+3z_2-z_3=34. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

The coefficients of z_3 are 0, 0, 0, -1, -1 respectively. These being multiplied into the respective equations, and the results added together, we obtain

$$-z_1-z_2+2z_3=-11. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

Solving these three normal equations (1), (2), and (3), we obtain the most probable values

$$\begin{cases} z_1=10{\cdot}375 \text{ ft.} \\ z_2=17{\cdot}625 \text{ ft.} \\ z_3=\ 8{\cdot}500 \text{ ft.} \end{cases}$$

Substituting these values of the elevations in the observation equations, we obtain

	v	v^2
$10{\cdot}375-10=$	$+{\cdot}375$	$\frac{1}{64}\times 9$
$17{\cdot}625-10\,375-\ 7=$	$+{\cdot}25$	,, 4
$17{\cdot}625\qquad\quad -18=$	$-{\cdot}375$	,, 9
$17{\cdot}625-8{\cdot}5\ -\ 9=$	$+{\cdot}125$	,, 1
$10{\cdot}375-8{\cdot}5\ -\ 2=$	$-{\cdot}125$	,, 1
		$\Sigma v^2=\frac{3}{8}$.

Example. Two quantities x and y are related to each other by a law of the form $y=Ax+C$, A and C being constants to be determined by experiments on x and y. Four observations were made :—

$$x=1, \quad 3, \quad 6, \quad 10;$$

$$y=4{\cdot}5, \quad 5{\cdot}9, \quad 7{\cdot}5, \quad 10{\cdot}2.$$

Find the constants A and C.

The observation equations are

$$4{\cdot}5 = A+C,$$

$$5{\cdot}9 = 3A+C,$$

$$7{\cdot}5 = 6A+C,$$

$$10{\cdot}2 = 10A+C.$$

The coefficients of C in all these equations being unity, one normal equation is obtained by adding the above equations together, thus

$$28{\cdot}1 = 20A+4C. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

The other normal equation is found as follows :—

$$1\times 4{\cdot}5 = 1^2A+1C,$$

$$3\times 5{\cdot}9 = 3^2A+3C,$$

$$6\times 7{\cdot}5 = 6^2A+6C,$$

$$10\times 10{\cdot}2 = 10^2A+10C,$$

$$169{\cdot}2 = 146A+20C. \quad \ldots\ldots\ldots\ldots \quad (2)$$

Solving these two equations, we obtain $A=0{\cdot}6239$ and $C=3{\cdot}906$. Hence we have

$$y=0{\cdot}624x+3{\cdot}91. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

The calculated values and the observed values may be compared as in the following table :—

x	y measured	y calculated by (3)	Difference v
1	4·5	4·53	+·03
3	5·9	5·78	−·12
6	7·5	7·65	+·15
10	10·2	10·15	−·05

Formation of Normal Equations from Observations of Unequal Weight.

Using the notations previously given, let there be observation equations of the following form :—

$$Z_1 = a_1 z_1 + b_1 z_2 + c_1 z_3 \text{ with weight } p_1,$$

$$Z_2 = a_2 z_1 + b_2 z_2 + c_2 z_3 \quad ,, \quad ,, \quad p_2,$$

.....

$$Z_n = a_n z_1 + b_n z_2 + c_n z_3 \quad ,, \quad ,, \quad p_n.$$

Since by (16) Σpv^2 is to be a minimum, we have

$$\begin{aligned} \frac{d}{dz_1}\Sigma pv^2 &= \Sigma 2pv\frac{dv}{dz_1} \\ &= \Sigma 2p(az_1 + bz_2 + cz_3 - Z)a \\ &= 0. \end{aligned}$$

Hence

$$\left.\begin{aligned} \Sigma paZ &= \Sigma pa^2.z_1 + \Sigma pab.z_2 + \Sigma pac.z_3. \\ \text{Similarly} \\ \Sigma pbZ &= \Sigma pab.z_1 + \Sigma pb^2.z_2 + \Sigma pbc.z_3, \\ \text{and} \\ \Sigma pcZ &= \Sigma pac.z_1 + \Sigma pbc.z_2 + \Sigma pc^2.z_3. \end{aligned}\right\} \quad \ldots\ldots \quad (18)$$

Example. Two quantities x and y are connected by the relation $y=Ax+C$, A and C being constants. The following observation equations were found :—

$$4{\cdot}5=2A+C \text{ with weight } 3,$$

$$6{\cdot}3=5A+C \quad ,, \quad ,, \quad 4,$$

$$9{\cdot}5=9A+C \quad ,, \quad ,, \quad 5.$$

Find the most probable values of A and C.

The normal equations are found as follows :—

$$3\times 4{\cdot}5 = 3\times 2A+3C$$

$$4\times 6{\cdot}3=4\times 5A+4C$$

$$5\times 9{\cdot}5=5\times 9A+5C$$

$$86{\cdot}2= \ 71A+12C$$

$$3\times 2\times 4{\cdot}5=3\times 2^2A+3\times 2C$$

$$4\times 5\times 6{\cdot}3=4\times 5^2A+4\times 5C$$

$$5\times 9\times 9{\cdot}5=5\times 9^2A+5\times 9C$$

$$580{\cdot}5= \ 517A+ \ 71C$$

Solution of these equations gives $A=0{\cdot}7273$ and $C=2{\cdot}880$. The equation expressing the relation between x and y therefore becomes

$$y=0{\cdot}727x+2{\cdot}88.$$

x	y observed	y calculated	v difference	p weight	pv^2
2	4·5	4·33	−·17	3	·0867
5	6·3	6·52	+·22	4	·1936
9	9·5	9·42	−·08	5	·0320

$$\Sigma pv^2=\cdot 3123$$

Example. It has been found by a set of experiments that two quantities x and y satisfy the equation $y=0{\cdot}4x+3$ fairly accurately for values of x from 1 to 6; and also by another set of equally good experiments that the equation $y=0{\cdot}42x+2{\cdot}9$ is satisfied fairly accurately for values of x from 4 to 9. It is required to find the most probable values of A and C in the equation $y=Ax+C$, which will be satisfied approximately for the whole range of x from 1 to 9.

The observation equations are

$$0{\cdot}4x+3=Ax+C \quad \text{for all values of } x \text{ from 1 to 6,}$$

and

$$0{\cdot}42x+2{\cdot}9=Ax+C \quad \text{for all values of } x \text{ from 4 to 9.}$$

The coefficient of C being unity, one normal equation is the sum of the above equations, thus:—

$$\int_1^6(0{\cdot}4x+3)dx+\int_4^9(0{\cdot}42x+2{\cdot}9)dx=\int_1^6(Ax+C)dx+\int_4^9(Ax+C)dx.$$

Multiplying all the observation equations by the coefficient x of the quantity A, we obtain the other normal equation, thus:—

$$\int_1^6(0{\cdot}4x^2+3x)dx+\int_4^9(0{\cdot}42x^2+2{\cdot}9x)dx=\int_1^6(Ax^2+Cx)dx+\int_4^9(Ax^2+Cx)dx.$$

Now

$$\int_1^6(0{\cdot}4x+3)dx=22,$$

$$\int_4^9(0{\cdot}42x+2{\cdot}9)dx=28{\cdot}15,$$

$$\int_1^6(Ax+C)dx=17{\cdot}5A+5C,$$

$$\int_4^9(Ax+C)dx=32{\cdot}5A+5C,$$

$$\int_1^6(0{\cdot}4x^2+3x)dx=81{\cdot}17,$$

$$\int_4^9 (\cdot 42x^2 + 2\cdot 9x)dx = 187\cdot 35,$$

$$\int_1^6 (Ax^2 + Cx)dx = \frac{1}{3} \times 215A + 17\cdot 5C,$$

and

$$\int_4^9 (Ax^2 + Cx)dx = \frac{1}{3} \times 665A + 32\cdot 5C.$$

Substituting these values of the integrals, we obtain the normal equations simplified

$$50\cdot 15 = 50A + 10C,$$

and

$$268\cdot 517 = \frac{880}{3}A + 50C.$$

From these are obtained $A = 0\cdot 4100$ and $C = 2\cdot 965$, so that the required equation stands, thus

$$y = 0\cdot 410x + 2\cdot 97.$$

x	y $\cdot 4x+3$	y $\cdot 42x+2\cdot 9$	y $\cdot 410x+2\cdot 97$
1	3·40	3·32	3·38
2	3·80	3·74	3·79
3	4·20	4·16	4·20
4	4·60	4·58	4·61
5	5·00	5·00	5·02
6	5·40	5·42	5·43
7	5·80	5·84	5·84
8	6·20	6·26	6·25
9	6·60	6·68	6·66

From this table it is seen that the single equation found by the Method of Least Squares is a very good approximation to the value of y for the whole range of the value of x.

Conditioned Observations. Observations are said to be conditioned when they are required to satisfy a certain condition or conditions exactly, and not approximately. Thus the successive bearings from one another of several objects as seen from a station completely round it are under the condition that the sum of the bearings must be exactly equal to four right angles. A general case is as follows:—Let there be n sets of simultaneous observations upon functions of p unknown quantities which are related to one another by q exact equations, p being greater than q and n greater than $p-q$, then the approximate values of the p quantities derived by the Method of Least Squares from the n observations must satisfy the q conditions exactly.

Example. Let the three angles of a triangle measured with equal care be a_1, a_2, a_3 degrees. Denoting the angles by z_1, z_2, z_3, since $z_1+z_2+z_3$ must be exactly equal to 180°, we have the following observation equations:—

$$z_1=a_1,$$

$$z_2=a_2,$$

$$z_1+z_2=180-a_3.$$

The normal equations are then

$$2z_1+z_2=a_1+180-a_3$$

and

$$z_1+2z_2=a_2+180-a_3$$

The solution of these two equations gives

$$z_1=a_1+\frac{1}{3}(180-a_1-a_2-a_3)$$

$$z_2=a_2+\frac{1}{3}(180-a_1-a_2-a_3)$$

and

$$z_3=a_3+\frac{1}{3}(180-a_1-a_2-a_3).$$

Example. Required to find an approximate value of $\sqrt{1+x}$ in the form of $C+Ax$, for any value of x from zero to $\frac{1}{2}$, with the condition that the root is exactly equal to 1 when x is zero.

Here it is evident that C must be equal to 1 and the observation equations are

$$\sqrt{1+x}=1+Ax$$

for all values of x from 0 to $\frac{1}{2}$. Hence the normal equation is

$$\int_0^{\frac{1}{2}} x\sqrt{1+x}\,dx=\int_0^{\frac{1}{2}} x\,dx+\int_0^{\frac{1}{2}} Ax^2\,dx,$$

$$\frac{2}{15}(3x-2)(1+x)^{\frac{3}{2}}=\frac{1}{2}x^2+\frac{1}{3}x^3A.$$

Taking the integrals between $x=0$ and $x=\frac{1}{2}$, we obtain $A=\cdot 4606$. Hence the required approximation to the square root is

$$\sqrt{1+x}=1+\cdot 46x.$$

Example. The measurements of three different diameters and the circumference of a circle gave a_1, a_2, a_3, and c ft. respectively. Find the diameter.

Denoting the diameter by z, the observation equations are

$$z=a_1,\quad z=a_2,\quad z=a_3,\quad \pi z=c.$$

The normal equation is therefore

$$3z+\pi^2 z=a_1+a_2+a_3+\pi c,$$

from which we obtain the diameter

$$z=\frac{a_1+a_2+a_3+\pi c}{3+\pi^2}.$$

Example. Required to approximate to $\tan\theta$ by means of $A\sin\theta$ for any value of θ from zero to a small angle α.

The observation equations are

$$A \sin \theta = \tan \theta$$

for all values of θ from zero to α. The normal equation is therefore

$$\int_0^\alpha A \sin^2\theta \, d\theta = \int_0^\alpha \sin \theta \tan \theta \, d\theta,$$

from which the coefficient A can be found,

$$A\left(\frac{\alpha}{2} - \frac{1}{4}\sin 2\alpha\right) = \log_e \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) - \sin \alpha.$$

Thus if α be 8 degrees, the coefficient A will be 1·00586 or approximately 1·006, so that

$$\tan \theta = 1{\cdot}006 \sin \theta.$$

Example. Two adjacent sides and a diagonal of a rectangle were found by measurements to be a_1, a_2, a_3 feet respectively. Assuming that the angles of the rectangle are exactly right angled, find the most probable values of the sides and diagonal.

Let z_1 and z_2 represent the most probable values of the lengths of the sides. The observation equations are then

$$z_1 = a_1, \quad z_2 = a_2, \quad \sqrt{z_1^2 + z_2^2} = a_3$$

The sum of squares of residual errors is

$$\Sigma v^2 = (z_1 - a_1)^2 + (z_2 - a_2)^2 + (\sqrt{z_1^2 + z_2^2} - a_3^2)^2.$$

This is to be a minimum and therefore we must have

$$\frac{d}{dz_1}\Sigma v^2 = 2(z_1 - a_1) + 2(\sqrt{z_1^2 + z_2^2} - a_3)\frac{z_1}{\sqrt{z_1^2 + z_2^2}} = 0$$

from which we have

$$(2z_1 - a_1)\sqrt{z_1^2 + z_2^2} = a_3 z_1.$$

Similarly

$$(2z_2-a_2)\sqrt{z_1^2+z_2^2}=a_3 z_2.$$

These are the normal equations, the solution of which gives

$$z_1=m a_1, \quad z_2=m a_2, \quad z_3=m\sqrt{a_1^2+a_2^2},$$

in which

$$m=\frac{1}{2}\left(1+\frac{a_3}{\sqrt{a_1^2+a_2^2}}\right).$$

Calculation of Probable Error. Expressing, by means of (5), the measures of precision h_1, h_2, h_3 ..... in terms of the probable errors r_1, r_2, r_3..... and substituting them in (14), we obtain

$$p_1 r_1^2=p_2 r_2^2=p_3 r_3^2=.....=r^2, \quad ..... \quad (19)$$

in which r is the probable error of an observation of weight unity.

Case I. Probable Error of the Arithmetical Mean of Direct Observations. The weight of the arithmetical mean of n equally good direct observations made on the same quantity is n, the weight of each being unity. Let r be the probable error of a single observation and r_0 that of the arithmetical mean. Then by (19) we have

$$r_0=\frac{r}{\sqrt{n}}. \quad ..... \quad (20)$$

Now to find r_1 the probable error of a single observation, proceed as follows:—The probability of the concurrence of the independent errors x_1, x_2, x_3,.....x_n is, as has been given previously,

$$P=\frac{(dx)^n}{h^n(\pi)^{\frac{n}{2}}}e^{-\frac{\Sigma x^2}{h^2}}$$

$$P=\left(\frac{(dx)^2}{\pi\Sigma x^2}\right)^{\frac{n}{2}} z^{\frac{n}{2}} e^{-z},$$

where $$z=\frac{\Sigma x^2}{h^2}.$$

This becomes zero when z is zero and also when z is infinite, that is when h is infinite and also when h is zero. There exists, therefore, some value of h that makes the probability P a maximum. Now for a given system of errors, it must be conceded that the observations have been as precise as possible and consequently h, which is proportional to the probable error, should have such a value as will make P a maximum. To find the value of h from the condition of maximum P, we have

$$\begin{aligned}\frac{dP}{dh}&=\frac{dP}{dz}\frac{dz}{dh}\\&=\text{const.}\left(\frac{n}{2}z^{\frac{n}{2}-1}e^{-z}-z^{\frac{n}{2}}e^{-z}\right)\left(-\frac{2}{h^3}\Sigma x^2\right)\\&=0,\end{aligned}$$

from which we have

$$\frac{n}{2}-z=0,$$

that is $$\frac{\Sigma x^2}{n}=\frac{h^2}{2}.$$

Therefore $$h=\sqrt{\frac{2\Sigma x^2}{n}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (21)$$

By (5), this becomes

$$\begin{aligned}r&=\cdot4769\sqrt{\frac{2\Sigma x^2}{n}}\\&=\cdot6745\sqrt{\frac{\Sigma x^2}{n}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (22)\end{aligned}$$

In a very large number of observations the errors closely agree with the residuals, and Σx^2 may be taken equal to Σv^2. But for a limited number of errors, Σv^2 is less than Σx^2, because the first is the minimum value of the second, and the relation between the two quantities may be written thus

$$\Sigma x^2 = \Sigma v^2 + \Delta.$$

The term Δ decreases with the increase of n, becoming zero when n is infinite, and becoming equal to Σx^2 when n is 1, because when there is but one observation, the observed value itself is the most probable value and therefore the residual must be zero. For the same value of n, Δ will increase or decrease with the value of Σx^2, because the more accurate the observation, the nearer will become Σv^2 to Σx^2. The exact value of Δ cannot be found and as the best approximation it may be taken as equal to $\frac{\Sigma x^2}{n}$. Then

$$\Sigma x^2 = \Sigma v^2 + \frac{\Sigma x^2}{n},$$

and

$$\frac{\Sigma x^2}{n} = \frac{\Sigma v^2}{n-1}.$$

Inserting this in (21) and (22) we have

$$h = \sqrt{\frac{2\Sigma v^2}{n-1}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (23)$$

$$r = \cdot 6745\sqrt{\frac{\Sigma v^2}{n-1}}. \quad \ldots\ldots\ldots\ldots\ldots \quad (24)$$

Hence by (20) we have

$$r_0 = \cdot 6745\sqrt{\frac{\Sigma v^2}{n(n-1)}}, \quad \ldots\ldots\ldots\ldots\ldots \quad (25)$$

which is the usual form of the formula for the probable error of the arithmetical mean of n observations.

Example. Show that the probable error of the arithmetical mean of two equally good observations is ·337 × (the difference between the two observations).

Case 2. Probable Error of the General Mean of Direct Observations of Unequal Weights. The weight of the general mean of direct observations having weights p_1, p_2, p_3,.....p_n is Σp. Using the notations previously given, the probable error of the general mean is

$$r_0 = \frac{r}{\sqrt{\Sigma p}}. \qquad \text{.....} \qquad (20')$$

To find r, the probable error of a single observation of weight unity, for the case of direct observations of unequal weights, the following process may be employed.

Writing h for the measure of precision of an observation of weight unity, we have by (14),

$$\frac{1}{h_1^2} = \frac{p_1}{h^2}, \qquad \frac{1}{h_2^2} = \frac{p_2}{h^2}, \qquad \frac{1}{h_3^2} = \frac{p_3}{h^2}, \qquad \text{.....}$$

The probability of the concurrence of the errors x_1, x_2, x_3,.....x_n is

$$P = \frac{(dx)^n}{h_1 h_2 \ldots\ldots h_n (\pi)^{\frac{n}{2}}} e^{-\left(\frac{x_1^2}{h_1^2} + \frac{x_2^2}{h_2^2} + \ldots\ldots + \frac{x_n^2}{h_n^2}\right)}$$

$$= \frac{(p_1 p_2 p_3 \ldots\ldots p_n)^{\frac{1}{2}} (dx)^n}{h^n (\pi)^{\frac{n}{2}}} e^{-\frac{1}{h^2}\left(p_1 x_1^2 + p_2 x_2^2 + \ldots\ldots + p_n x_n^2\right)}$$

$$= \left(p_1 p_2 p_3 \ldots\ldots p_n\right)^{\frac{1}{2}} \left(\frac{(dx)^2}{\pi \Sigma p x^2}\right)^{\frac{n}{2}} z^{\frac{n}{2}} e^{-z},$$

where $$z = \frac{\Sigma p x^2}{h^2}.$$

As the condition of maximum value of P, a result similar to (21) is obtained; thus

$$h=\sqrt{\frac{2\Sigma px^2}{n}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (21')$$

and by (5), this becomes

$$r=\cdot 6745\sqrt{\frac{\Sigma px^2}{n}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (22')$$

Now the relative magnitudes of Σpx^2 and Σpv^2 being considered the same as in Case 1, the above two equations become

$$h=\sqrt{\frac{2\Sigma pv^2}{n-1}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (23')$$

and

$$r=\cdot 6745\sqrt{\frac{\Sigma pv^2}{n-1}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (24')$$

This last equation combined with (20') gives

$$r_0=\cdot 6745\sqrt{\frac{\Sigma pv^2}{(n-1)\Sigma p}}. \quad \ldots\ldots\ldots\ldots\ldots \quad (25')$$

Case 3. Probable Errors for Independent Observations made upon a Function of two or more Unknowns. The relative magnitudes of Σpx^2 and Σpv^2 may be expressed by the equation

$$\Sigma px^2=\Sigma pv^2+\Delta,$$

in which, first, Δ is positive, because Σpv^2 is the minimum value of Σpx^2; secondly, Δ decreases as n increases, becoming zero when n is infinite and becoming equal to Σpx^2 when there is but one observation, because in the case of one observation only, the observed value itself is the most probable value and therefore the residual must be zero; thirdly, for a given number of observations, Δ increases when Σpx^2 increases, because the more accurate the observations, the nearer will be Σpv^2 to Σpx^2; and fourthly,

Δ increases when the number of unknown quantities, q, increases, because when q is one, Δ must be equal to $\frac{\Sigma px^2}{n}$ of Case 2 and when q is equal to n, the condition is the same as that of a single observation on one unknown, and Δ must be equal to Σpx^2. To sum up, we have

Δ is positive;

$\Delta = \Sigma px^2$ when n is equal to 1;

$\Delta = 0$,, n is infinite;

$\Delta = \frac{\Sigma px^2}{n}$,, q is equal to 1;

$\Delta = \Sigma px^2$,, q ,, ,, ,, n.

Hence $$\Sigma px^2 = \Sigma pv^2 + \frac{q\Sigma px^2}{n},$$

therefore $$\frac{\Sigma px^2}{n} = \frac{\Sigma pv^2}{n-q},$$

in which q is the number of unknown quantities z_1, z_2, z_3,..... Hence by (5), (21), and the above result, we obtain the probable error of each of Z_1, Z_2, Z_3,..... reduced to weight unity,

$$r = \cdot 6745\sqrt{\frac{\Sigma pv^2}{n-q}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (24'')$$

and also

$$h = \sqrt{\frac{2\Sigma pv^2}{n-q}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (23'')$$

The Expressions for the unknown Quantities z_1, z_2, z_3,.....z_q. The general form of the normal equations (18) shows that the result is the same as if each observation equation had been first multiplied by the square root of its weight, by which means it would be reduced to weight unity, and the system would take the form (17), so that a now stands for $a\sqrt{p}$, b for $b\sqrt{p}$, c for $c\sqrt{p}$, &c.

Find the following determinants, whose elements are the terms on the first member of (17) and the coefficients of the unknown quantities $z_1, z_2, z_3, \ldots\ldots z_q$,

$$D = \begin{vmatrix} \Sigma aa & \Sigma ab & \Sigma ac & \ldots\ldots & \Sigma al \\ \Sigma ba & \Sigma bb & \Sigma bc & \ldots\ldots & \Sigma bl \\ \Sigma ca & \Sigma cb & \Sigma cc & \ldots\ldots & \Sigma cl \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots \\ \Sigma la & \Sigma lb & \Sigma lc & \ldots\ldots & \Sigma ll \end{vmatrix}$$

All the 2nd. factors a of the products, which form the elements of the 1st. column of the determinant D, replaced by Z,

$$D_{z1} = \begin{vmatrix} \Sigma aZ & \Sigma ab & \Sigma ac & \ldots\ldots & \Sigma al \\ \Sigma bZ & \Sigma bb & \Sigma bc & \ldots\ldots & \Sigma bl \\ \Sigma cZ & \Sigma cb & \Sigma cc & \ldots\ldots & \Sigma cl \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots \\ \Sigma lZ & \Sigma lb & \Sigma lc & \ldots\ldots & \Sigma ll \end{vmatrix}$$

All factors b in the 2nd. column similarly replaced by Z,

$$D_{z2} = \begin{vmatrix} \Sigma aa & \Sigma aZ & \Sigma ac & \ldots\ldots & \Sigma al \\ \Sigma ba & \Sigma bZ & \Sigma bc & \ldots\ldots & \Sigma bl \\ \Sigma ca & \Sigma cZ & \Sigma cc & \ldots\ldots & \Sigma cl \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots \\ \Sigma la & \Sigma lZ & \Sigma lc & \ldots\ldots & \Sigma ll \end{vmatrix}$$

All factors c in the 3rd. column similarly replaced by Z,

$$D_{z3} = \begin{vmatrix} \Sigma aa & \Sigma ab & \Sigma aZ & \ldots\ldots & \Sigma al \\ \Sigma ba & \Sigma bb & \Sigma bZ & \ldots\ldots & \Sigma bl \\ \Sigma ca & \Sigma cb & \Sigma cZ & \ldots\ldots & \Sigma cl \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots \\ \Sigma la & \Sigma lb & \Sigma lZ & \ldots\ldots & \Sigma ll \end{vmatrix}$$

&c. &c.

Then the general expressions for z_1, z_2, z_3,..... are

$$z_1 = \frac{D_{z1}}{D}, \qquad z_2 = \frac{D_{z2}}{D}, \qquad z_3 = \frac{D_{z3}}{D}, \qquad \ldots\ldots\ldots \quad (26)$$

Let z_1, z_2, z_3,.....z_q be expanded in terms of ΣaZ, ΣbZ, ΣcZ, ...ΣlZ, thus

$$\left.\begin{aligned} z_1 &= Q_1\Sigma aZ + Q_2\Sigma bZ + Q_3\Sigma cZ + \ldots\ldots\ldots + Q_q\Sigma lZ, \\ z_2 &= R_1\Sigma aZ + R_2\Sigma bZ + R_3\Sigma cZ + \ldots\ldots\ldots + R_q\Sigma lZ, \\ z_3 &= S_1\Sigma aZ + S_2\Sigma bZ + S_3\Sigma cZ + \ldots\ldots\ldots + S_q\Sigma lZ. \end{aligned}\right\} (27)$$

Now if D_{z1} be expanded in terms of the elements of its first column; D_{z2} in terms of the elements of its second column; and D_{z3} in terms of the elements of its third column; we have

$$D_{z1} = \Sigma aZ.D_{aa} + \Sigma bZ.D_{ab} + \Sigma cZ.D_{ac} + \ldots\ldots\ldots,$$

$$D_{z2} = \Sigma aZ.D_{ab} + \Sigma bZ.D_{bb} + \Sigma cZ.D_{bc} + \ldots\ldots\ldots,$$

$$D_{z3} = \Sigma aZ.D_{ac} + \Sigma bZ.D_{bc} + \Sigma cZ.D_{cc} + \ldots\ldots\ldots,$$

&c. &c.

in which

$$\left\{\begin{array}{llll} D_{aa}, & D_{ab}, & D_{ac}, & \ldots\ldots\ldots\ldots \\ D_{ab}, & D_{bb}, & D_{bc}, & \ldots\ldots\ldots\ldots \\ D_{ac}, & D_{bc}, & D_{cc}, & \ldots\ldots\ldots\ldots \end{array}\right.$$

are the first minors obtained by suppressing in D the row and column which contain the elements

$$\left\{\begin{array}{llll} \Sigma aa, & \Sigma ab, & \Sigma ac, & \ldots\ldots\ldots\ldots, \\ \Sigma ab, & \Sigma bb, & \Sigma bc, & \ldots\ldots\ldots\ldots, \\ \Sigma ac, & \Sigma bc, & \Sigma cc, & \ldots\ldots\ldots\ldots, \end{array}\right.$$

Comparing the coefficients Q, R, S in (27) with the coefficients in the above expansions and using (26), we get

$$Q_1=\frac{D_{aa}}{D}, \quad Q_2=\frac{D_{ab}}{D}, \quad Q_3=\frac{D_{ac}}{D}, \quad \ldots\ldots\ldots$$

$$R_1=\frac{D_{ab}}{D}, \quad R_2=\frac{D_{bb}}{D}, \quad R_3=\frac{D_{bc}}{D}, \quad \ldots\ldots\ldots$$

$$S_1=\frac{D_{ac}}{D}, \quad S_2=\frac{D_{bc}}{D}, \quad S_3=\frac{D_{cc}}{D}, \quad \ldots\ldots\ldots$$

&c. &c.

The Weights and Probable Errors of the Unknown Quantities $z_1, z_2, \ldots\ldots z_q$.

Let $z_1, z_2, z_3, \ldots\ldots$ be fully expanded in terms of the observed quantities, thus

$$\left.\begin{aligned} z_1&=\alpha_1 Z_1+\alpha_2 Z_2+\alpha_3 Z_3+\ldots\ldots+\alpha_n Z_n,\\ z_2&=\beta_1 Z_1+\beta_2 Z_2+\beta_3 Z_3+\ldots\ldots+\beta_n Z_n,\\ z_3&=\gamma_1 Z_1+\gamma_2 Z_2+\gamma_3 Z_3+\ldots\ldots+\gamma_n Z_n. \end{aligned}\right\} \quad (28)$$

Then if r_{z_1} denotes the probable error of z_1, and r that of a standard observation, that is, the common probable error of each of the observed values of $Z_1, Z_2, Z_3, \ldots\ldots Z_n$, we shall have by (8)

$$r_{z_1}^2=r^2\Sigma\alpha^2.$$

If p_{z_1} be the weight of the unknown quantity z_1, we have by (19)

$$p_{z_1} r_{z_1}^2=r^2,$$

and therefore $$p_{z_1}=\frac{1}{\Sigma\alpha^2}.$$

Similarly $$p_{z_2}=\frac{1}{\Sigma\beta^2},$$

and $$p_{z_3}=\frac{1}{\Sigma\gamma^2}.$$

Now to determine the values of these weights, we proceed as follows. Equating the coefficients of Z_1, Z_2, Z_3, ... in the first of (28) to those in the first of (27), we obtain

$$\left.\begin{aligned} a_1 &= a_1 Q_1 + b_1 Q_2 + c_1 Q_3 + \ldots\ldots + l_1 Q_q, \\ a_2 &= a_2 Q_1 + b_2 Q_2 + c_2 Q_3 + \ldots\ldots + l_2 Q_q, \\ &\ldots\ldots \qquad \ldots\ldots \qquad \ldots\ldots \\ a_n &= a_n Q_1 + b_n Q_2 + c_n Q_3 + \ldots\ldots + l_n Q_q. \end{aligned}\right\} \quad (29)$$

Multiplying the first of these equations by a_1, the second by a_2, and so on, and adding the results, we obtain

$$\Sigma a^2 = \Sigma aa.Q_1 + \Sigma ba.Q_2 + \Sigma ca.Q_3 + \ldots\ldots + \Sigma la.Q_q. \quad \ldots\ldots\ldots\ldots\ldots \quad (30)$$

Multiplying the first of (29) by a_1, the second by a_2, and so on, and adding, the result is

$$\begin{aligned} \Sigma aa &= \Sigma aa.Q_1 + \Sigma ab.Q_2 + \Sigma ac Q_3 + \ldots\ldots + \Sigma al.Q_q \\ &= \Sigma aa \frac{D_{aa}}{D} + \Sigma ab \frac{D_{ab}}{D} + \Sigma ac \frac{D_{ac}}{D} + \ldots\ldots + \Sigma al \frac{D_{al}}{D}, \\ &= \frac{1}{D}\left(\Sigma aa.D_{aa} + \Sigma ab.D_{ab} + \Sigma ac.D_{ac} + \ldots\ldots\right) \\ &= \frac{1}{D} \times D = 1. \end{aligned}$$

In the same way we have

$$\begin{aligned} \Sigma ba &= \Sigma ab.Q_1 + \Sigma bb.Q_2 + \Sigma bc.Q_3 + \ldots\ldots + \Sigma bl.Q_q \\ &= \Sigma ab.\frac{D_{aa}}{D} + \Sigma bb.\frac{D_{ab}}{D} + \Sigma bc.\frac{D_{ac}}{D} + \ldots\ldots \\ &= \frac{1}{D}\left(\Sigma ab.D_{aa} + \Sigma bb.D_{ab} + \Sigma bc.D_{ac} + \ldots\ldots\right), \end{aligned}$$

that is,

$$D\Sigma ba = \begin{vmatrix} \Sigma ab & \Sigma ab & \Sigma ac & \ldots\ldots & \Sigma al \\ \Sigma bb & \Sigma bb & \Sigma bc & \ldots\ldots & \Sigma bl \\ \Sigma cb & \Sigma cb & \Sigma cc & \ldots\ldots & \Sigma cl \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots \\ \Sigma lb & \Sigma lb & \Sigma lc & \ldots\ldots & \Sigma ll \end{vmatrix} = 0,$$

because the first two columns are identical. In the same way $\Sigma ca, \ldots\ldots \Sigma la$ are all zero.

Hence (30) becomes

$$\Sigma a^2 = Q_1, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (31)$$

and the expressions for the weights therefore become

$$\left.\begin{aligned} p_{z_1} &= \frac{1}{Q_1} = \frac{D}{D_{aa}}, \\ p_{z_2} &= \frac{1}{R_2} = \frac{D}{D_{bb}}, \\ p_{z_3} &= \frac{1}{S_3} = \frac{D}{D_{cc}}. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (32)$$

The probable errors of z_1, z_2, $z_3, \ldots\ldots$ may be expressed thus

$$\left.\begin{aligned} r_{z_1} &= \frac{r}{\sqrt{p_{z_1}}} = r\sqrt{Q_1}, \\ r_{z_2} &= \frac{r}{\sqrt{p_{z_2}}} = r\sqrt{R_2}, \\ r_{z_3} &= \frac{r}{\sqrt{p_{z_3}}} = r\sqrt{S_3}, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (33)$$

the value of r being given by (24″).

Example. Consider the normal equations previously given for the measurements of the elevations of three points above a given bench mark. They were

$$\begin{aligned} 3z_1 - z_2 - z_3 &= 5, \\ -z_1 + 3z_2 - z_3 &= 34, \\ -z_1 - z_2 + 2z_3 &= -11. \end{aligned}$$

As it is necessary to keep the second members of the normal equations in their literal forms, let A, B, C be written for 5, 34 and -11 respectively. The determinant which forms the denominators of all the coefficients of A, B, and C in the expansions of z_1, z_2, and z_3 is

$$D = \begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 4 & -1 & 0 \\ -4 & 3 & -4 \\ 0 & -1 & 3 \end{vmatrix} = \begin{vmatrix} 4 & -1 & 0 \\ 0 & 2 & -4 \\ 0 & -1 & 3 \end{vmatrix}$$

$$= 4 \begin{vmatrix} 2 & -4 \\ -1 & 3 \end{vmatrix} = 4(6-4) = 8.$$

$$D_{z1} = \begin{vmatrix} A & -1 & -1 \\ B & 3 & -1 \\ C & -1 & 2 \end{vmatrix} = \begin{vmatrix} A & -1 & 0 \\ B & 3 & -4 \\ C & -1 & 3 \end{vmatrix}$$

$$= A \begin{vmatrix} 3 & -4 \\ -1 & 3 \end{vmatrix} - B \begin{vmatrix} -1 & 0 \\ -1 & 3 \end{vmatrix} + C \begin{vmatrix} -1 & 0 \\ 3 & -4 \end{vmatrix} = 5A + 3B + 4C.$$

$$D_{z2} = \begin{vmatrix} 3 & A & -1 \\ -1 & B & -1 \\ -1 & C & 2 \end{vmatrix} = \begin{vmatrix} A & -3 & -1 \\ B & 1 & -1 \\ C & 1 & 2 \end{vmatrix}$$

$$= A \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} - B \begin{vmatrix} -3 & -1 \\ 1 & 2 \end{vmatrix} + C \begin{vmatrix} -3 & -1 \\ 1 & -1 \end{vmatrix} = 3A + 5B + 4C.$$

$$D_{z3} = \begin{vmatrix} 3 & -1 & A \\ -1 & 3 & B \\ -1 & -1 & C \end{vmatrix} = \begin{vmatrix} A & 3 & -1 \\ B & -1 & 3 \\ C & -1 & -1 \end{vmatrix} = \begin{vmatrix} A & 4 & -1 \\ B & -4 & 3 \\ C & 0 & -1 \end{vmatrix}$$

$$=4\begin{vmatrix} A & 1 & -1 \\ B & -1 & 3 \\ C & 0 & -1 \end{vmatrix} = 4\begin{vmatrix} A & 1 & 0 \\ B & -1 & 2 \\ C & 0 & -1 \end{vmatrix}$$

$$=4\left\{ A\begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} - B\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} + C\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} \right\}$$

$$=4(A+B+2C)=4A+4B+8C.$$

Thus we have

$$z_1=\frac{5}{8}A+\frac{3}{8}B+\frac{1}{2}C,$$

$$z_2=\frac{3}{8}A+\frac{5}{8}B+\frac{1}{2}C,$$

$$z_3=\frac{1}{2}A+\frac{1}{2}B+\quad C,$$

and the weights are therefore

$$p_{z_1}=\frac{8}{5}, \qquad p_{z_2}=\frac{8}{5}, \qquad p_{z_3}=1.$$

The probable error of each of the five observations 10, 7, 18, 9, and 2 ft. is by (24″)

$$r=\cdot 6745\sqrt{\frac{\Sigma pv^2}{n-q}}$$

$$=\cdot 6745\sqrt{\frac{\cdot 375}{5-3}}=\cdot 2921.$$

Then by (33), we have

$$r_{z_1}=r\sqrt{Q_1}=\cdot 2921\sqrt{\frac{5}{8}}=\cdot 2309,$$

$$r_{z_2}=r\sqrt{R_2}=\cdot 2921\sqrt{\frac{5}{8}}=\cdot 2309,$$

$$r_{z_3}=r\sqrt{S_3}=\cdot 2921\sqrt{1}=\cdot 2921.$$

The calculated values of z_1, z_2, and z_3 are therefore

$$z_1 = 10{\cdot}375 \pm {\cdot}231 \text{ ft.},$$

$$z_2 = 17{\cdot}625 \pm {\cdot}231 \text{ ft.},$$

$$z_3 = 8{\cdot}500 \pm {\cdot}292 \text{ ft.}$$

Probable Errors for Conditioned Observations. When conditioned observations are so adjusted that the q unknown quantities in the n observation equations are reduced to $q-n'$ independent quantities by means of n' conditional equations, the probable error of an observation of the weight unity is given by (24″), in which q is to be replaced by $q-n'$, or

$$r = {\cdot}6745\sqrt{\frac{\Sigma pv^2}{n-q+n'}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (34)$$

The probable errors and weights of z_1, z_2, z_3,..... are to be found exactly as in the previous case by (32) and (33).

The Rejection of Doubtful Observations. It sometimes happens that the magnitude of an observation appears to be too much out of proportion compared with the rest of the observations forming one series. For example, suppose that one observation Z in a system of a hundred observations has its residual error x equal to 4·5 times the probable error r of the mean of the hundred. The probability integral corresponding to $\frac{x}{r} = 4{\cdot}5$ is found from a Table to be ·9976, which is the probability of any error being less than $4{\cdot}5\,r$, and therefore the probability of any error being at least as large as $4{\cdot}5\,r$ is ·0024. The probable number of observations having greater errors than $4{\cdot}5\,r$ is therefore $100 \times {\cdot}0024$, that is 0·24. This being very much less than unity, the observation Z may be rejected as doubtful.

Chauvenet's criterion for the rejection of doubtful observations is as follows:—

Let x= the residual error of an observation Z,

r=the probable error of the mean of many observations of which Z is one,

n=the whole number of observations,

P=the probability corresponding to error x.

Then nP is the probable number of errors that are less than x and $n-Pn$ is the number of errors greater than x. The criterion is "*Reject the observation Z if $n-nP$ is less than $\frac{1}{2}$.*"

In the example above given, this number is ·24 and therefore by the criterion the observation may be rejected. Speaking generally, if the residual error of an observation is 4 or 5 times the probable error of the most probable value, that observation may be regarded as doubtful, and if moreover the number of observations is not very large, it may be rejected.

Relative Accidental Errors. Combination of Concordant Determinations. Suppose that two determinations Z_1 and Z_2 of a quantity whose true value is Z, have been made under circumstances exactly alike, so that they may be considered to have the same standard of precision. Let x_1 and x_2 be the errors of the determinations, then $Z_1=Z+x_1$, $Z_2=Z+x_2$, and $Z_1-Z_2=x_1-x_2$. The difference between the two results is therefore the algebraic difference between the accidental errors remaining in the two determinations; this may be called their *relative accidental error*. Now if Z_1 and Z_2 were considered as *independent measurements* of two quantities, and r_1 and r_2 their probable errors, then that of Z_1-Z_2 would be by (7) $r=\sqrt{r_1^2+r_2^2}$. Since the true value of Z_1-Z_2 is zero, it is seen that $r=\sqrt{r_1^2+r_2^2}$ is the probable error

for a system of errors, of which the relative error x_1-x_2 forms one constituent.

Now in combining any two determinations to obtain a single mean value, it is necessary first of all to form a fair judgement whether there exist or not in them any systematic errors produced by causes that are different in the two determinations. For this purpose the two quantities x_1-x_2 and r, discussed above, may be compared. If x_1-x_2 is small enough to be regarded as a relative accidental error, the two determinations Z_1 and Z_2 may safely be assumed to be free from different systematic errors, though there may be the same systematic errors in them. In this case the two determinations may be said to be *concordant* and they may be combined by means of (13′) to form a mean value, thus

$$\frac{\frac{Z_1}{r_1^2}+\frac{Z_2}{r_2^2}}{\frac{1}{r_1^2}+\frac{1}{r_2^2}},*$$

and the probable error R of this mean value is given by

$$\frac{1}{R^2}=\frac{1}{r_1^2}+\frac{1}{r_2^2}.$$

Example. A certain length has been determined by a method as 4336 ft. $\pm$3·4 ft. and a second determination by another method is 4324 ft. $\pm$13·6 ft. Are these concordant? and if so, find the most probable mean value.

In this case, $r_1=3{\cdot}4$, $r_2=13{\cdot}6$, and $x_1-x_2=12$, so that $r=\sqrt{r_1^2+r_2^2}=\sqrt{(3{\cdot}4)^2+(13{\cdot}6)^2}=14{\cdot}0$, being greater than the rela-

* This may be written thus

$$Z_1-\frac{r_1^2}{r_1^2+r_2^2}\left(Z_1-Z_2\right) \quad \text{or} \quad Z_2+\frac{r_2^2}{r_1^2+r_2^2}\left(Z_1-Z_2\right).$$

tive accidental error. It is thus quite reasonable that there should be an error as large as 12. In fact, from a Probability Table, the value of probability for the ratio $\frac{x}{r}=\frac{12}{14}=0{\cdot}857$ is ·437 and therefore the probability that an error be equal to or greater than 12 is ·563. Combining the two determinations by means of (13′), we have the most probable mean value

$$4336-\frac{3{\cdot}4^2}{(3{\cdot}4)^2+(13{\cdot}6)^2}\times 12$$

$$=4336-\frac{12}{17}=4335{\cdot}29 \text{ ft.},$$

and
$$R=\frac{r_1\, r_2}{\sqrt{r_1^2+r_2^2}}=\frac{3{\cdot}4\times 13{\cdot}6}{\sqrt{(3{\cdot}4)^2+(13{\cdot}6)^2}}=3{\cdot}300.$$

The final result is

$$4335{\cdot}3 \text{ ft.}\pm 3{\cdot}3 \text{ ft.}$$

Combination of Discordant Determinations. If, on the other hand, the difference between two determinations is very large compared with the relative probable error r, it will be found from a Table of Probability Integral that the probability of the relative accidental error being as large as x_1-x_2, is only a very small fraction, which will not be likely to be true. We are then justified in assuming that the difference x_1-x_2 is mainly due to some causes peculiar to each determination. In this case the two determinations are said to be *discordant*. Supposing that there is no special reason for preferring either of them to the other, the final result should be the simple arithmetical mean. The probable error of this mean is to be found by (25), which, for the case of two observations only, becomes ·337 × (the difference between the two observations).

Example. Suppose that one determination of a certain length is $347{\cdot}5^{mm.} \pm {\cdot}4^{mm.}$ and another is $344{\cdot}7^{mm.} \pm {\cdot}6^{mm.}$.

Here $r_1 = {\cdot}4$, $r_2 = {\cdot}6$, $x_1 - x_2 = 2{\cdot}8$, so that $r = \sqrt{{\cdot}4^2 + {\cdot}6^2} = {\cdot}721$, $\frac{x}{r} = \frac{x_1 - x_2}{r} = \frac{2{\cdot}8}{{\cdot}721} = 3{\cdot}88$. The probability integral corresponding to this ratio $= {\cdot}9911$, $1 - {\cdot}9911 = {\cdot}0089 = \frac{1}{112}$.

The last fraction is the probability that the relative accidental error should be as large as $2{\cdot}8^{mm.}$, which is not likely to be the case, unless the number of observations be very large. The final determination should therfore be the arithmetical mean, which is $347{\cdot}1^{mm.}$, the probable error being ${\cdot}337 \times 2{\cdot}8 = {\cdot}9^{mm.}$.

Example. The length of a meter bar was measured at temperatures $20°C$, $40°C$, $50°C$, $60°C$, giving observed lengths $1000{\cdot}22$, $1000{\cdot}65$, $1000{\cdot}90$, $1001{\cdot}05^{mm.}$ respectively. Find the length at $0°C$ and the expansion per degree [Prof. Johnson's Theory of Errors and Method of Least Squares, page 117.] Find also the temperature at which the bar is exactly one meter in length.

Let the length of the bar at $0°C$ be $1000 + z_1$ and the expansion per degree be z_2. The observation equations are then

$$\begin{aligned} z_1 + 20\ z_2 &= {\cdot}22, \\ z_1 + 40\ z_2 &= {\cdot}65, \\ z_1 + 50\ z_2 &= {\cdot}90, \\ z_1 + 60\ z_2 &= 1{\cdot}05. \end{aligned}$$

The normal equations are

$$\begin{aligned} 4z_1 + 170\ z_2 &= 2{\cdot}82, \\ 170z_1 + 8100\ z_2 &= 138{\cdot}4. \end{aligned}$$

Let A and B be written for $2{\cdot}82$ and $138{\cdot}4$ respectively. Solving the normal equations, we obtain

$$z_1 = \frac{81}{35} A - \frac{68}{350} B = -\cdot 196,$$

$$z_2 = \frac{4}{3500} B - \frac{17}{350} A = \cdot 0212,$$

in which

$$Q_1 = \frac{81}{35} \quad \text{and} \quad R_2 = \frac{4}{3500}.$$

Now

$$\begin{aligned}
v_1 &= -\cdot 196 + 20 \times \cdot 0212 - \cdot 22 = \cdot 008, & v_1^2 &= \cdot 000064,\\
v_2 &= -\cdot 196 + 40 \times \cdot 0212 - 65 = \cdot 002, & v_2^2 &= \cdot 000004,\\
v_3 &= -\cdot 196 + 50 \times \cdot 0212 - \cdot 90 = -\cdot 036, & v_3^2 &= \cdot 001296,\\
v_4 &= -\cdot 196 + 60 \times \cdot 0212 - 1\cdot 05 = \cdot 026, & v_4^2 &= \cdot 000676.\\
& & \Sigma v^2 &= \cdot 002040.
\end{aligned}$$

$n-q=4-2=2$, and the probable error of any of the four measurements is

$$r = \cdot 6745 \sqrt{\frac{\Sigma v^2}{n-q}}$$

$$= \cdot 6745 \sqrt{\frac{\cdot 002040}{2}}$$

$$= \cdot 02154.$$

$$r_{z_1} = r\sqrt{Q_1} = \cdot 02154 \sqrt{\frac{81}{35}} = \cdot 03277,$$

$$r_{z_2} = r\sqrt{R_2} = \cdot 02154 \sqrt{\frac{4}{3500}} = \cdot 0006797.$$

The length of the bar at any temperature $t°C$ is given by

$$l = 1000 - 0\cdot 196 + \cdot 0212\, t.$$

Hence the bar is exactly one meter in length at the temperature given by

$$t = \frac{\cdot 196}{\cdot 0212} = 9\cdot 245°C.$$

Resumé:—The length at $0°C = 999{\cdot}804^{mm.} \pm {\cdot}033$, expansion per degree $C = {\cdot}0212 \pm {\cdot}0007$, and the temperature at which the bar is just one meter in length $= 9{\cdot}25°C$.

Empirical Formulæ. When a rational law expressing the relation to one another of observed quantities is not known, an approximate empirical formula may be assumed to represent the law of variation within the range of observation. The constants in such a formula are best determined by the application of the Method of Least Squares.

The first step in the establishment of an emperical formula is to plot on squared paper the given observations as abscissæ and ordinates and to draw by inspection a mean curve for the points so plotted. A large class of curves may be represented by parabolas, thus

$$y = A + Bx + Cx^2 + Dx^3 + \ldots\ldots,$$

in which A, B, C, &c. represent constants whose values are to be determined from the observations. When the successive differences of y of the second and higher orders, for a given finite difference of x, diminish very slowly, it will be necessary to keep four or more terms on the right hand side of the equation. In such a case, the following form of the equation is very convenient,

$$\log y = A + Bx + Cx^2 + Dx^3 + \ldots\ldots,$$

which will cause the successive differences to diminish very rapidly.

When the quantity represented by y tends to a definite and constant value with the increase of x, the following equation may be used:—

$$y = \frac{A + Bx + Cx^2 + \ldots\ldots + Hx^n}{a + bx + cx^2 + \ldots\ldots + hx^n}.$$

Particular cases of this equation are sometimes useful,

$$y=\frac{Bx}{a+bx}, \qquad y=\frac{A}{a+cx^2}.$$

The last equation is in the form of Gordon-Rankine formula for long columns. When the coefficient H of the highest power of x in the numerator of the above general equation is zero, the value of y continually approaches zero as x increases indefinitely.

Another class of equations often useful especially in connection with the properties of gases is

$$y+b=c(x+a)^n,$$

in which a, b, c, and also n may be determined from observations. The case in which a and b are zero is well known. The equation is then reducible to a straight line form by taking the logarithms of both sides of the equation; thus,

$$\log y=n\log x+\log c.$$

When the quantity y passes through repeating cycles, another class of equations representing a large variety of phenomena may be used,

$$\begin{aligned} y=A&+B_1\sin\theta+B_2\cos\theta\\ &+C_1\sin 2\theta+C_2\cos 2\theta\\ &+D_1\sin 3\theta+D_2\cos 3\theta\\ &+\quad \&c. \qquad \&c. \end{aligned}$$

in which $\theta=\frac{\pi x}{a}$, a being the period or periodic time of the independent variable x, considered as time. As instances of phenomena performing repeating cycles, may be mentioned the annual variation of the atmospheric temperature, changes of barometric pressure, the ebb and flow of the tide, the motion of

the steam valve worked by any radial valve gear in a steam engine, &c.

Many other forms of equations can easily be proposed, such as for example

$$y = A + \frac{B}{x+a} + \frac{C}{(x+a)^2} + \frac{D}{(x+a)^3} + \ldots\ldots,$$

$$y = ke^{-mx} \sin ax, \qquad \&c. \qquad \&c.$$

What form of equation will agree best with a given system of observations, is a problem that can only be solved by trial. Assuming a plausible equation, find out the values of the constants involved in it by the Method of Least Squares. Now calculate by means of the resulting equation the values corresponding to the observations, and if the agreement is not so close as is desirable, another equation should be proposed and tried.

A few examples showing the **application of the Method of Least Squares** to engineering problems are here added.

Example. A large series of tests on the strength of full size columns of timber were carried out under the direction of Prof. Lanza, Massachusetts Institute of Technology, and for posts of white pine the following average values of the strength constants were found :—

$l \div h$	0 to 10	10 to 35	35 to 45	45 to 60
p	2500	2000	1500	1000

All the posts being flat at the ends and rectangular in section, of breadth b and depth h, $l \div h$ in the table is the ratio of the length to the least side, and p is the breaking load in lbs. per

sq. inch of the section. Required to find the values of the constants in Gordon-Rankine formula for this kind of timber.

Solution:—For a rectangular section $k^2=\frac{I}{A}=\frac{1}{12}bh^3\div bh=\frac{1}{12}h^2$, and Gordon-Rankine formula takes the following form, for columns with ends flat,

$$p=\frac{f}{1+12\,a\left(\frac{l}{h}\right)^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

For shortness write $a=\frac{A}{1000}$ and $f=1000B$. The above equation then becomes

$$p=\frac{1000\,B}{1+\frac{12\,A}{1000}\left(\frac{l}{h}\right)^2},$$

which may be written thus:—

$$\frac{p}{1000}+\frac{12p}{(1000)^2}\left(\frac{l}{h}\right)^2A-B=0. \quad \ldots\ldots \quad (2)$$

Now in forming observation equations from (2), the value of $\left(\frac{l}{h}\right)^2$ corresponding to each p, should be the mean square of the ratio $\frac{l}{h}$, that is $\frac{1}{3}(x_2^2+x_2x_1+x_1^2)$, where x_2 and x_1 represents the limits of the ratio $\frac{l}{h}$ within which the p is intended to be used. Thus for

$$p=2500, \quad \left(\frac{l}{h}\right)^2=\frac{1}{3}\left(10^2+10\times0+0^2\right)=\frac{1}{3}\times100,$$

$$\text{„}=2000, \quad \text{„} \quad =\frac{1}{3}\left(35^2+35\times10+10^2\right)=\frac{1}{3}\times1675,$$

$$\text{„}=1500, \quad \text{„} \quad =\frac{1}{3}\left(45^2+45\times35+35^2\right)=\frac{1}{3}\times4825,$$

$$\text{„}=1000, \quad \text{„} \quad =\frac{1}{3}\left(60^2+60\times45+45^2\right)=\frac{1}{3}\times8325.$$

The corresponding values of $\frac{12p}{1000} \cdot \frac{\left(\frac{l}{h}\right)^2}{1000}$ are

for $p=2500$, $\frac{12\times 2500}{1000} \times \frac{100}{3\times 1000} = 1.00$,

„ $=2000$, $\frac{12\times 2000}{1000} \times \frac{1675}{3\times 1000} = 13{\cdot}40$,

„ $=1500$, $\frac{12\times 1500}{1000} \times \frac{4825}{3\times 1000} = 28{\cdot}95$,

„ $=1000$, $\frac{12\times 1000}{1000} \times \frac{8325}{3\times 1000} = 33{\cdot}30$.

Substitution of these in the equation (2) gives the observation equations,

$$2{\cdot}5 + 1{\cdot}00\,A - B = 0,$$
$$2{\cdot}0 + 13{\cdot}40\,A - B = 0,$$
$$1{\cdot}5 + 28{\cdot}95\,A - B = 0,$$
$$1{\cdot}0 + 33{\cdot}30\,A - B = 0.$$

From these, the normal equations are obtained in the usual manner,

$$7{\cdot}000 + 76{\cdot}65\,A - 4{\cdot}00\,B = 0,$$

and

$$106{\cdot}025 + 2127{\cdot}55\,A - 76{\cdot}65\,B = 0,$$

from which the constants A and B are found to be

$$A = {\cdot}04268 \quad \text{and} \quad B = 2{\cdot}568,$$

so that

$$a = \frac{A}{1000} = \frac{{\cdot}04268}{1000} = \frac{1}{23430},$$

and $f = 1000\,B = 2568$ lbs. per sq. inch. Finally the formula becomes

$$p = \frac{2570}{1 + \frac{12}{23400}\left(\frac{l}{h}\right)^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

The values of p calculated by this formula may be compared with the averages found from the tests.

$\left(\frac{l}{h}\right)^2$	$\left(\frac{l}{h}\right)$	Average p from tests	Calculated p by (3)	Difference of p's	Percent diff.
$\frac{1}{3}\times 100$	5·77	2500	2527	+27	+1·1
$\frac{1}{3}\times 1675$	23·63	2000	1998	− 2	−0·1
$\frac{1}{3}\times 4825$	40·10	1500	1408	−92	−6·1
$\frac{1}{3}\times 8325$	52·68	1000	1061	+61	+6·1

Example. The variation in the amount of coal consumed per indicated horse power per hour by the same typed steam engines are due mainly to the variation of the load. A small engine when exerting 12, 45, and 60 indicated horse power has been found to burn 36, 18, and $8\frac{1}{2}$ lbs. of coal per hour per indicated horse power in their ordinary factory use under altering loads. The maximum indicated power of the engine was 60. From these data make out a formula of the following form :—

$$C = A + B \times (\text{indicated } H.P.),$$

where C represents the amount of coal in lbs. per indicated horse power per hour, A and B constants to be determined.

Solution :—The observation equations are

$$36 = A + 12\,B,$$
$$18 = A + 45\,B,$$
$$8\tfrac{1}{2} = A + 60\,B.$$

The normal equations are then

$$62\tfrac{1}{2} = 3\,A + 117\,B,$$
$$1752 = 117\,A + 5769\,B.$$

From these we obtain $A=43{\cdot}0$ and $B=-0{\cdot}565$, so that

$$C=43{\cdot}0-0{\cdot}565\ I.H.P.,$$

which gives 36·2, 17·8, and 9·1 lbs. respectively for the three cases stated in the question.

Example. Given the following values of the volume v in cub. ft. per lb. weight of saturated steam at absolute pressure p lbs. per sq. inch, it is required to find the two constants n and c in the empirical formula

$$pv^n=c.$$

Pressure p	60	180	300
Volume v	7·024	2·493	1·535

Solution :—The above equation may be written in the form

$$\log p+n\log v=\log c.$$

The normal equations are

$$\Sigma\log p+n\Sigma\log v=N\log c,$$

$$\Sigma(\log p)(\log v)+n\Sigma(\log v)^2=(\log c)\Sigma\log v,$$

where N is the number of observation equations, being 3 in this problem. Now from the data, the following numbers are to be calculated.

p	v	$\log p$	$\log v$	$(\log v)^2$	$(\log p)\times(\log v)$
60	7·024	1·77815	0·84658	0·716698	1·50535
180	2·493	2·25527	0·39672	0·157387	0·89471
300	1·535	2·47712	0·18611	0·034637	0·46102
Σ		6·51054	1·42941	0·908722	2·86108

Substituting these numerical values of the coefficients in the normal equations, we obtain

$$6{\cdot}51054+1{\cdot}42941\,n=3\log c,$$

$$2{\cdot}86108+0{\cdot}908722\,n=1{\cdot}42941\log c.$$

From these equations, the constants are found to be

$$n=1{\cdot}0586,\quad c=\log^{-1}2{\cdot}67459=472{\cdot}7.$$

Hence the equation required is

$$pv^{1{\cdot}0586}=472{\cdot}7$$

or

$$\log p+1{\cdot}0586\log v=2{\cdot}67459.$$

The result of calculation by this approximate formula for saturation curve may be compared with the values of v given in the steam table.

p	v by Table	v by Formula	Error of v by Formula	Error in percentage
60	7·024	7·028	+·004	+0·06
120	3·656	3·651	−·005	−0·14
180	2·493	2·493	−·003	−0·12
240	1·898	1·897	−·001	−0·05
300	1·535	1·537	+·002	+0·13

Example. The following measurements were made on a compound condensing steam engine. Find a relation between the indicated and brake horse power in a linear form.

Indicated *H. P.*	288	223	136
Brake *H. P.*	249	189	108.

Ans. $B=0{\cdot}927\,I-17{\cdot}9,$

or $I=1{\cdot}079\,B+19{\cdot}3.$

The brake power calculated by the first of these equations are 249·1, 188·8, and 108·2 corresponding to the three sets of measurements, to which the agreement is very close.

Example. By a series of tests of a certain steam engine, the following results were obtainad.

Condensing		Non-condensing	
Indicated *H. P.*	Brake *H. P.*	Indicated *H. P.*	Brake *H. P.*
50·5	40	42·5	35
38·5	30	31	25
29	20	24	17·5
17	10	15·5	10
8	0	5·5	0

The full power of the engine when working as a condensing or non-condensing engine corresponds to the highest figure in the respective column in the table. Required to find a relation between the brake and indicated horse power in a linear form.

Ans. $B = 0{\cdot}939\, I - 6{\cdot}85$ or $I = 1{\cdot}065\, B + 7{\cdot}3$ } condensing.

$B = 0{\cdot}949\, I - 5{\cdot}00$ or $I = 1{\cdot}0535\, B + 5{\cdot}265$ } non-condensing.

The results of calculation may be compared with the measurements.

Condensing			Non-condensing		
B	I by test	I calculated	B	I by test	I calculated
40	50·5	49·9	35	42·5	42·1
30	38·5	39·3	25	31	31·6
20	29	28·6	17·5	24	23·7
10	17	18·0	10	15·5	15·8
0	8	7·3	0	5·5	5·3

Example. From a number of power tests made by Captain Sankey on a Willans' engine capable of developing 150 indicated horse power, the following may be quoted :—

Indicated *H. P.*	104·5	49	12·5
Brake *H. P.*	94·5	38	0

From these data the following relation between the indicated and brake horse power may be obtained :—

$$B = 1{\cdot}027\, I - 12{\cdot}7.$$

Comparison of the Formula with the Test.

I	B by Test	B by Formula	Error of B by Formula
104·5	94·5	94·7	+0·2
49	38	37·7	−0·3
12·5	0	0·2	+0·2

Taking the above formula as an approximation to truth, the amount of power lost in friction and also the percentage loss in friction may be exhibited in a tabular form as follows :—

Willans' Engine.

Indicated horse power I	Brake horse power B	$I-B$	$100\times\frac{I-B}{I}$
150	141·4	8·6	5·7
100	90·0	10·0	10·0
75	64·3	10·7	14·3
50	38·7	11·3	22·6
25	13·0	12·0	48·0
20	7·9	13·1	65·5

Here the amount of power lost in friction at a large load is actually smaller than that at a less load. This is in contradiction to the usually accepted law of friction in all machines, and shows the importance of power trials of engines.

Tokyo, May, 1910.

XXXI.

Disposition of Boiler Feed Pipes.

[*The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. XIV, No.* 24, *February,* 1911.]

In Cornish and Lancashire boilers or in other boilers of similar type, a simple and effective method of obtaining good circulation of water is to extend the feed pipe into the inside of the boiler, carrying it longitudinally along one side from the front end to nearly the back end of the boiler. This inside feed pipe has a completely plugged end and in its last 4, 5 or 6 ft. of length a number of small holes are drilled, of about $\frac{1}{4}''$ diameter, all these holes being arranged in one line and pointing obliquely towards the bottom of the boiler. By this device the feeding of the boiler is effected at a comparatively cool portion near the back end, in the form of many small parallel jets, which are distributed over a large area and directed towards the bottom of the boiler. The effect of these parallel jets is to produce an induced current of water in the circumferential direction. The pitch of the small holes drilled in the feed pipe is fitly made to diminish towards the plugged end and the combined area of the holes for a $2\frac{1}{2}''$ pipe may be about $\frac{1}{3}$ the area of the pipe.

Now the extra pressure required by the feed pump in forcing the water through the small holes may be estimated in the

following manner. Consider the case of a $2\frac{1}{2}''$ feed pipe. The mean velocity of feed water through the pipe may be assumed at about $3\frac{1}{2}$ ft. per second. Let the combined area of the holes be $\frac{1}{3}$ the area of the pipe and assume that the coefficient of contraction of the jets is 0·64. The velocity of the jets is then $3\frac{1}{2} \div (\frac{1}{3} \times \cdot 64) = 16\cdot 4$ ft. per second. The head due to this velocity is 4·19 ft. of water, which is equal to 1·81 lbs. per sq. inch. The coefficient of resistance of the jets through the holes may be taken at 0·06, so that the extra pressure required by the feed pump is $1\cdot 06 \times 1\cdot 81 = 1\cdot 92$, that is, about 2 lbs. per sq. inch. For a feed pipe of any size a small extra pressure on the feed pump, of a few lbs. per sq. inch, may first be assumed and the corresponding combined area of the holes may be found by calculation.

The writer had an opportunity, over a year ago, of applying the disposition of feed pipe described above to an existing Cornish boiler 4′ 6″ in diameter and 17′ 0″ in length. No drop of steam pressure due to the supply of comparatively cold feed water has since been experienced and steadier and easier working of the boiler has been the result.

Tokyo, April, 1910.

XXXII.

The Relation between the Horse Power and the Weight of an Engine.

[*The Journal of the Society of Mechanical Engineers, Tokyo, Japan, Vol. XV, No. 26, February,* 1912.]

The stress at a point in a section of a machine working at a steady speed is made up of two parts; one part is the stress due to the useful working force transmitted through the section and the other that due to the accelerating force arising from the reciprocation or rotation of the mass of machine parts. The power exerted at any instant by a reciprocating engine is equal to the product of the total useful stress transmitted through a cross section of the working part and the velocity of that section. Divide the mean value of the power in ft. lbs. per sec. for one complete working cycle of the engine by 550 ft. lbs. per sec.; and the quotient is the so-called horse power of the engine. The accelerating force due to the reciprocation varies conjointly as the mass of the moving part, the linear dimension of the reciprocation or the stroke, and the square of the number of revolutions per unit time. Since the sum of the useful stress and the acceleration stress must necessarily be limited by the safe stress of the material, there exists for a high speed engine

of a given type a certain definite speed of rotation for which the horse power is a maximum. The weight of an engine evidently varies as the cube of linear dimension; and it may be shown that the weight per horse power of a high speed engine increases as the linear dimension increases. These and like points may be discussed in the following manner.

Let

P=the useful load transmitted through a cross section of the working part of the engine, as for example the total initial pressure on the piston, transmitted through the piston rod, and crank;

n=the number of revolutions per unit time;

p=the stress due to P on a particular member such as the piston rod;

p_0=the stress due to the accelerating force on the same member;

l=the length of the same member or of any other member; for an engine l may conveniently be taken as the stroke of the piston;

A=the area of the section, at a point of which the stresses p and p_0 act in the same direction;

ρ=the density of the material of which the engine is made.

The stress due to the acceleration or retardation of the reciprocating parts is expressed by

$$p_0 = \text{const.} \times \frac{l \times \text{mass} \times n^2}{A}.$$

But the weight or mass of the engine may be expressed by

$$M = \text{const.}\ \rho l A = c\rho l^3, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

where c is a constant depending on the type of the engine. Hence

$$p_0 = \text{const.}\ \rho n^2 l^2 = k_0 \rho n^2 l^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

where k_0 is a constant depending on the mode of distribution of the mass of the engine parts. It will be the smaller the better the design of the engine. Since $n\,l$ is proportional to the linear velocity, the above equation shows that the stress due to acceleration varies as the square of the piston speed. The same relation also holds for the stress due to centrifugal force in a uniformly rotating part such as the rim of a flywheel, it being expressed by $p_0 = \rho v^2$, where v is the linear volecity of the rim.

The stress due to the useful load on the engine is given by

$$p = k\frac{P}{l^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

where k is another constant depending on the mode of action of the load P and also on the form of the engine parts.

Now the safe stress of the material of which the engine is made is given by

$$f = p + p_0 = k\frac{P}{l^2} + k_0 \rho n^2 l^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

so that

$$P = \frac{1}{k}(f l^2 - k_0 \rho n^2 l^4)\,; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

but the horse power of the engine is

$$HP = \text{const.}\ nl.\ P. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

Therefore $$HP = C(f n l^3 - k_0 \rho n^3 l^5). \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

Now considering the speed of rotation and the size of the engine to be quantities independent of each other, the expression above for the horse power becomes maximum in two different cases.

Case 1. Considering engines of different sizes, all running at a given number of revolutions per min., the horse power becomes maximum when

$$\frac{dHP}{dl}=C(3fnl^2-5k_0\rho n^3l^4)=0,$$

that is, when

$$\frac{3}{5}f=k_0\rho n^2l^2=p_0. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

The corresponding maximum power is

$$maximum\ HP=\frac{2}{5}Cfnl^3. \quad \ldots\ldots\ldots\ldots\ldots \quad (9)$$

Since nl is proportional to the linear velocity we may write $nl=k'v$, where k' is a constant and the above becomes

$$maximum\ HP=\frac{2}{5}Ck'fl^2v. \quad \ldots\ldots\ldots\ldots\ldots \quad (10)$$

From (8) it is seen that for maximum power at a given speed of rotation, the engine should be so designed that the stress due to the accelerating force shall be 3/5 th of the working stress of the material, the remaining 2/5 th only being that due to the useful working load. Eliminating by means of (8) the speed of rotation n from the equation (9), we obtain

$$maximum\ HP=\frac{2}{5}\left(\frac{3}{5k_0}\right)^{\frac{1}{2}}C\frac{f^{1\frac{1}{2}}l^2}{\rho^{\frac{1}{2}}}, \quad \ldots\ldots\ldots \quad (11)$$

which is of the same form as the equation arrived at by Mr. Lanchester by the principle of dimension equation.

Case 2. Considering engines of the same dimensions running at different speeds, that which develops the maximum power is given by

$$\frac{dHP}{dn} = C(fl^3 - 3k_0\rho n^2 l^5) = 0,$$

that is, by

$$\frac{1}{3}f = k_0\rho n^2 l^2 = p_0. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

The corresponding maximum power is

$$maximum\ HP = \frac{2}{3}Cfnl^3, \quad \ldots\ldots\ldots\ldots\ldots \quad (13)$$

which differs from (9) only in the coefficient. In this case the engine should be so designed that the inertia stress shall be 1/3 rd of the safe working stress, the remaining 2/3 rds being that due to the useful stress. Eliminating by means of (12) the speed of rotation n from the equation (13), we obtain

$$maximum\ HP = \frac{2}{3}\left(\frac{1}{3k_0}\right)^{\frac{1}{2}} C.\frac{f^{1\frac{1}{2}}l^2}{\rho^{\frac{1}{2}}}, \quad \ldots\ldots \quad (14)$$

a result of the same form as the equation (11) obtained in Case 1. The ratio of the maximum horse power given by (13) to that given by (9) is $1\frac{2}{3}$, on the supposition that the engines in the two cases are exactly similar. For in the two cases the intensities of useful stresses are limited to $\frac{2}{5}$ and $\frac{}{3}$ of the safe working stress of the material.

As a problem occurring in actual practice it is however better to state the above result in the following modified form. Suppose that the same amount of horse powers is developed by each of two high speed engines, one being designed for maximum horse power at a given speed of rotation and the other for maximum horse power with a given size and that both engines have the same principal dimensions such as the diameter of cylinder and the stroke of the piston. Then since the total use-

ful stress transmitted through a corresponding cross section must be the same in the two engines, we must have

$$\frac{2}{5}fA_1=\frac{2}{3}fA_2,$$

where A_1 and A_2 are the values of A for the two engines. Hence

$$A_1=1\frac{2}{3}A_2,$$

and this will be true for all the working parts through which the useful stress is transmitted. Thus the first engine is $1\frac{2}{3}$ times as heavy as the second engine. Dividing (1) by (7), we obtain the mass or weight per horse power,

$$\frac{M}{H}=\frac{c\rho}{Cn(f-k_0\rho n^2l^2)}. \quad \ldots\ldots\ldots\ldots\ldots \quad (15)$$

This becomes a minimum when

$$\frac{1}{3}f=k_0\rho n^2l^2,$$

which is just the same as the equation (12). Thus the condition of maximum horse power expressed by (12) satisfies also the condition of minimum weight per horse power. Combining (15) with (12), we obtain

$$\frac{M}{H}=\frac{3c}{2C}(3k_0)^{\frac{1}{2}}l\left(\frac{\rho}{f}\right)^{1\frac{1}{2}}. \quad \ldots\ldots\ldots\ldots\ldots \quad (16)$$

Hence supposing that the engines are so designed and worked at such a speed that the stress due to the accelerating force is $\frac{1}{3}$rd of the working stress of the material, the weight per horse power will be proportional to the linear measurement, or in other words, the horse power developed per lb. weight of engines of similar type decreases in the inverse proportion of the linear dimension. The larger is the size of an engine, the greater is

the total horse power in the duplicate proportion as shown by (11) or (14), but at the same time the smaller becomes the horse power per lb. weight. Thus arises the following problem. It is required to obtain a certain amount of total horse power with the least weight of the engine. By (14) it is seen that an engine having four equal cylinders can develop just the same horse power as a single cylinder engine of double the linear measurement. But by (1) the weight of the four cylinder engine is

$$M=4\times c\rho\left(\frac{1}{2}l\right)^3=\frac{1}{2}c\rho l^3,$$

that is, one half the weight of the single cylinder engine. In general a single cylinder engine and a similar engine having N equal cylinders each $\frac{1}{\sqrt{N}}$th in linear measurement can develop the same amount of horse power. This follows at once from equation (14); for,

$$maximum\ HP=\text{const.}\times N\left(\frac{l}{N^{\frac{1}{2}}}\right)^2=\text{const.}\times l^2.$$

But as regards the total weight of the second engine, we have by (1)

$$M=Nc\rho\left(\frac{l}{\sqrt{N}}\right)^3=\frac{1}{\sqrt{N}}(c\rho l^3), \quad \ldots\ldots\ldots \quad (17)$$

so that *the weight decreases in the inverse proportion of the square root of the number of cylinders.* This is a broad governing principle relating to the weight of an engine of a given power. Against it there is however a set back that the efficiency of an engine decreases as the size of the cylinder becomes smaller. In aeronautical engines the weight of the fuel to be carried is thus increased, so that the multiplication of cylinders will be limited by this factor.

Now to examine the materials used for the working parts of an engine. Let f_1 and ρ_1 be the tensile strength and the density of one kind of material, and f_2 and ρ_2 the corresponding quantities for another kind of material. So far as the horse power and weight of an engine are concerned, it is seen from (14) that these materials will be equally good, if

$$f_1^{1\frac{1}{2}}\rho_1^{-\frac{1}{2}}=f_2^{1\frac{1}{2}}\rho_2^{-\frac{1}{2}}$$

that is, if

$$f_2=f_1\left(\frac{\rho_2}{\rho_1}\right)^{\frac{1}{3}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (18)$$

For example, commercial rolled aluminium weighing about $\rho_2=\cdot096$ lb. per cubic inch may be compared with mild steel of average quality used for the construction of steam boilers. The latter metal weighs about $\rho_1=0\cdot283$ lb. per cubic inch and has an average tensile strength of about $f_1=28$ tons per square inch. Hence the rolled aluminium would be as good as the boiler steel, if

$$f_2=28\left(\frac{\cdot096}{\cdot283}\right)^{\frac{1}{3}}$$

$$=19\cdot5 \text{ tons per sq. inch.}$$

Rolled aluminium commercially pure has a tensile strength ranging from about $6\frac{1}{2}$ to $10\frac{1}{2}$ tons per square inch, while the hard rolled plates of aluminium alloy prepared by Mr. Yarrow with the addition of 6 per cent. copper and adopted for the construction of a torpedo boat gave 14 to 16 tons per square inch tensile strength. These are not sufficiently high to equal the boiler steel. An alloy of aluminium termed "Duraluminium," invented recently and manufactured by the Electric and Ordnance Accessories Co. of Birmingham is said to contain upwards of

90 per cent. of aluminium, and to have a specific gravity of about 2·8 and a melting point of 650° *C.* The makers state that the tensile strength of the alloy is as high as 40 tons per square inch when the metal is hard and has a little elongation; 28 to 30 tons per square inch with 15 per cent. elongation in 2 inch gauge length; and 25 tons per square inch with 20 per cent. elongation in 2 inch length. Specific gravity of 2·8 is equivalent to ·101 lb. per cubic inch, so that Duraluminium will be better than the boiler steel, if its tensile strength is higher than

$$f_2 = 28\left(\frac{\cdot 101}{\cdot 283}\right)^{\frac{1}{3}} = 19\cdot 9 \text{ tons per sq inch,}$$

so that this metal in its weakest and most ductile state is much superior to the boiler steel, so far as the horse power and weight of the engine are concerned.

It may be added here that in the adoption of a new material for the construction of an engine or structure, the resistance of it to shearing would probably be another very important quality demanding our careful examination.

INDEX 索 引

Numbers refer to pages; those in bold type to pages in Japanese

Accidental errors, relative, 411
Actual breaking resilience, 175
Aluminium, rolled, 436
Approximate value of integral of a continuous function, 95
——— ——— — the differential coefficient, 106
Approximation, Newtons method of, 81
Arithmetical mean, 382
——— ———, probable error of, 397
Axiom, **121**

バイスバック (ワイスバッハ), **19**
バッケット, **37**
Barlow's Tables of Squares, Cubes, &c., 115
Barré, L. A., 304
Beam, strain energy of, 173
—— of uniform rectangular section, 310
Beams, strength of curved, 347
Bending test, universal repetitive, 180
——— — of cast iron, 179
——— — — timbers, 178
Bernoulli's theorem, 183
Best velocity in the spiral casing, 263
微積分, **112**
Blowers, pressure, 45
———, Roots', 45
Board of Trade Rule, 300
ボイデン, **8**
Boyden, **8**
Boiler feed pipe, 427
Bolt and nut, strength of, 64
Box girder, 312
Brake and indicated H. P., 424
Briggs' Second Stage Mathematics, **126**
Bucket, pump, **37**

Cable, suspended, considered as similar structures, 314
Calculation of probable error, 397
Cast iron, bending test of, 179
—— ——, torsion test of, 134
Castle's, F., Elementary Practical Mathematics, **126**
Cauchy, 304
Centrifugal pump, friction in, 246
——— ——, tests of a forced vortex, 281
——— ——— having divergent vortex chamber, 181
——— pumps, sizes of, 156
——— ——, comparision and discussion, 232
——— ——, general theory of, 193
Centroid, graphic method of finding, 357
Chauvenet's criterion, 411
Circular disk flywheel, 318
Circulation of hot water, velocity of, 25
Clearance in steam engine cylinders, 28
Coal consumed per I. H. P., 421
Coefficient of discharge from a rectangular notch, 134
——— — reliability, 176
Collapsing, 69
Column, solid cylindrical, 312

Column of brickwork, 312
——— — uniform strength, 320
Comparison and discussion of centrifugal pumps, 232
Compound condensing steam engine, horse power of, 423
Compression of steam, 28
Concordant determinations, 411
Conditioned observations, 394
——— ———, probable errors for, 410
Cooling, rate of, 25
Coupling, flexible, 305
——— rod, 318
Crank pin effort, 53
Current, divergent radiating, 186
———, radiating, 181
Curved beams, strength of, 347
Cylinder clearance, 28

Defective speed regulation, 133
Delivery of centrifugal pump, 218
Determinations, concordant, 411
———, discordant, 413
De Morgan, Prof. A., 116
Differential and integral calculus, **112**
——— coefficient, approximate value of, 106
——— ——— of a function, 110
Discordant determinations, 413
Discussion of centrifugal pumps, 232
Divergent free circular vortex, 186
——— — spiral vortex, 187
——— radiating current, 186
——— vortex chamber, 212, 242
Doubling plate, 136
Doubtful observations, rejection of, 410
Draught, furnace, **64**
Draw hooks, strength of, 360
Duraluminium, 436

Efficiency of centrifugal pump, 218, 238
Empiricial formulæ, 416
Engine, horse power of compound condensing steam, 423
———, horse power and weight of an, 429
ENGINEER, THE, 28, 37
ENGINEERING, 140
Engineering articles, ready made, 153
English Barracks Improvement Commissioners, 19
Error, probable, 372, 397
——, propagation of, 373
Errors, relative accidental, 411
—— of observation, 365
Euler formula, 69, 320
Euler-Maclaurin formula, 118
Experiment on the strength of bolt and nut, 64

Fan, 75
フェーヤベルン, **3, 18**
Fairbairn, **3, 18**
Feed pipes, boiler, 427
Flexible coupling, 305
Flour mill, **28**
Flywheel, **44**, 311
———, circular disk, 318
———, high speed, 316
———, theory of, 122
フォンテン, **8**
Fontain, **8**
Forced draught, **65**
—— vortex, 188, 212
—— —— pump, 216
Formation of normal equations, 385, 390
Formulæ, empirical, 416
——— for helical springs, 297
Foundation footing, 343
フォルニーロン, **8**
Fourneyron, **8**
——— turbines at Niagara, 123, 129
Free circular vortex, 182

Free spiral vortex, 184
—— —— ——, divergent, 187
Friction in centrifugal pumps, 246
——— — divergent vortex centrifugal pumps, 279
——— — ——— —— chamber, 254
——— — non-divergent vortex chamber, 252
——— — spiral casing, 259
——— — vortex chamber, 275
——— through impeller, 258
普通の數學順序, **109**

General mean, probable errors of, 400
——— theory of centrifugal pumps, 193
Girder, steel box, 312
Gordon, Lewis, 333
——— formula for a long tube, 70
Gordon-Rankine formula, 70, 321, 419
Graphic method of finding centroid and moment of inertia, 357
——— ——— — —— modified area, 355
Grashof, 334
Guide vanes, 212, 213
—— ——, straight, 280

Hartnell's experiments on helical spring, 299
はやさ、速度, **56**
はずみ輪, **44**
Head of water, **3**
High speed flywheel, 316
挽粉, **28**
比例, **122**
Hodgkinson, 340
Hood, 17, 18
Hooks, stresses in, 356
Horizontal selfcontained short stroke engines, sizes of, 153
Horse power and weight of an engine, 429
—— —— of a chimney, **72**
Hotel management, **114**
Hydraulic lifting jacks, sizes of, 158
Hypothetical breaking resilience, 175

Impeller vanes, straight, 280
Inertia of moving parts of an engine, 53
Independent observations, probable error of, 401
Indicated and brake H. P., 424, 425
Induced draught, **65**
ねのくち剪斷試驗片, **87**
Inokuty's shearing block, **87**
Integral, probability, 371
Interpolation, 72
———, Lagrange's theorem for, 92
——— formula, 103
——— ———, Newton's, 92
圍壇裡の通氣法, **64**

Japanese water wheels, **27**
ジョンバル, **8**
Jonval, **8**
Journal of the College of Engineering, Tokyo Imperial University, 163, 173, 180, 181, 281
——— — — Engineering Society, Japan, **27, 37, 64**; 6, 16, 320, 343
——— — — Society of Mechanical Engineers, Tokyo, Japan, 38, 45, 64, 69, 122, 137, 145, 153, 181, 281, 297, 305, 307, 347, 427, 429

計算尺, **119**
Kelvin, Lord, Nat. Phil., 304, 368
煙出しの馬力, **72**
機械工學術語集, **60**
工學會誌, **27, 37, 64**
工學叢誌, **1**
工業的數學, **108**

米脊, **28**
公理, **121**
汲揚及壓揚喞筒, **37**

Lagrange's theorem for interpolation, 92
Lanchester, 432
Lanza, Prof., 418
Laplace, 100
Least squares, method of, 365
—— ———, principle of, 381
Lévy, 69, 71
Life insurance, 80
Lilly, Prof. W. E., 340
Lloyd's Rule for flat plates, 136, 144
Logarithmic table, use of, **124**

眞野文二, 教授, **58, 59**
Mano, Prof. B., **58, 59**
Mean, arithmetical, 382
Method of least squares, 365
Mild steel, relation of strength to weight of, 436
Modified area, 351
——— ——, graphic method of finding, 355
木材の剪斷試驗, **86**
Moment of inertia, graphic method of finding, 357
文部省實業學務局報告, **79, 108**
Morin, amount of ventilation allowed by, 20
Morley, Prof. A., 351
Mortality and life insurance, 80
Motion of water particle, 269

Natural draught, **65**
Newton's interpolation formula, 92
——— method of approximation, 81
日本水車の說, **27**
Normal equations, 385, 390
Number of cylinders and weight, 435
Numbers, conception of quantity expressed by, **116**
Nut, strength of bolt and, 64

Observation, errors of, 365
Observations, conditioned, 394
———, doubtful, 410
———, weights of, 383
Oldham coupling, 305
Ordinary method of teaching mathematics, **109**
Ormsby, M. T., **81, 126**
壓揚及汲揚喞筒, **37**
押し込み通氣法, **65**
Overshot wheel, **9**

Parkes on ventilation, 19
Peabody's Table of Saturated Steam, 88
Pelton wheel, 130
Perry, Prof. J., **80**
Perry's Applied Mechanics, 304
Pettenkofer on ventilation, 20
ピストン, **40**
Piston, **40**
—— and crank pin effort, 53
Plunger, **37**
——— and bucket pump, **51**
——— — piston pump, **47**
ポンセレット(ポンスレー), **8**
Poncelet, **8**
Posts of white pine, strength of, 418
Practical mathematics, **79**
Pressure blowers, 45
——— gauges, tests of, 180
——— head in vortex, 189
Principle of least squares, 381
Probability, 366
——— curve, 370
——— integral, 371
Probable error, 372, 397
——— —— of arithmetical mean, 397

Probable errors for conditioned observations, 410
——— —— — independent observations, 401
——— —— of general mean, 400
——— —— — unknown quantities, 405
Propagation of error, 373
——— — pulsation, 145
Pumps, sizes of centrifugal, 156
———, lift and force, **37**
——, rotary, 45
ブランヂャル, **37**
———、バケット喞筒, **51**
———、ピストン喞筒, **47**

Rachet screw jacks, sizes of, 157
Radiating current, 181
——— ———, divergent, 186
ランキン教授, **3**
Rankine, Prof., **3**, 299, 333
Rankine's Useful Rules and Tables, 299
Ratio, **122**
Ready made engineering articles, 153
Rectangular notch, coefficient of discharge, 134
Rejection of doubtful observations, 410
Relative accidental errors, 411
Reliability, coefficient of, 176
Replogle's mechanical relay governor, 123
Report of College of Engineering, Tokyo Imperial University, 134, 135
—— — Earthquake Investigation Committee, **58, 59, 86**
—— on strength of timbers, **58, 59**
Resilience, 175
Rice dressing, **28**
Riehlé Brothers' hydraulic testing machine, 67
RIGAKU KYOKWAI ZASSHI, 1
Rites, F. M., on fly wheel, 129
Rolled aluminium, 436
Roots' pressure blower, 45
Rotary pumps, 45

Sankey on Willans' engine, 425
Saturated steam, volume and weight of, 422
Saxelby's A Course in Practical Mathematics, **126**
Screw jacks, sizes of, 157
扇風機, **75**
Shearing test of timbers, **86**, 163
Shibaura Engineering Works, Tokyo, forced vortex pump made at, 216, 281
震災豫防調査會報告, **58, 59, 86**
自然通氣法, **65**
Similar structures, 307
Simpson's first rule, 99
——— rule, extension of, 6
Size of ready made engineering articles, 153
Slide rule, **119**
Solid cylindrical column, 312
Solids, volume of certain, 1
Speed of centrifugal pump at which delivery commences, 217
—— regulation, defective, 133
Spiral casing, 200, 208
Springs, formulæ for helical, 297
——— of rectangular section, 303
Square steel springs, 300
Steel, mild, relation of strength to weight of, 436
Strain energy of a beam, 173
Strength of bolt and nut, 64
——— — curved beams, 347
——— — draw hooks, 360
——— — flat part of a steam boiler fitted with a doubling plate, 136
——— — foundation footing, 343
——— — long tubes under external pressure, 69

Strength of posts of white pine, 418
——— — timbers, report on, **58, 59**
——— — wheel teeth, 38
Stress due to acceleration of reciprocating parts, 430
—— — — useful load, 431
Stresses in hooks, 356
——— — similar machines, 316
Successive difference, 160
——— differences of a continuous function, 86
Sudden application of the load on the prime mover, 133
——— change of velocity, variation of pressure due to, 191
——— enlargement of section, loss of head due to, 190
——— removal of the load on the prime mover, 133
——— stoppage of water in a pipe, 145
吸出し通氣法, **65**
水車意匠法, **1**
水車の說、日本, **27**
數量の說明, **116**
Suspended cables considered as similar structures, 314

對數表, **124**
高さ、水の, **3**
Takemura, K. G., 347
田邊朔郎, 教授, **58, 59**
Tanabe, Prof. S., **58, 59**
田中不二, **86**
Tanaka, Fuji, **86,** 134, 173, 180
Tandem compound engines, sizes of, 155
Tests of a forced vortex centrifugal pump, 281
—— — pressure gauges, 180
—— — vacuum gauges, 135
Theory of centrifugal pumps, 181
——— — ——— ——, general, 193
——— — difference, 72
Theory of flywheel, 122
Timbers, bending test of, 178
———, shearing test of, **86,** 163
Time, duration and instant of, **125**
Tortion test of cast iron, 134
Transaction of Japanese Institution of Naval Architects, 53
トレッドゴルド, **19**
Tredgold, **19,** 17, 333
Tubes under external pressure, 69

Universal repetitive bending test, 180
Unwin, Prof. W. C., 38, 44, 334
Utimaru S., 134, 135

Vacuum gauges, tests of, 135
Velocity, the term in Japanese, **56**
——— in the spiral casing, best, 263
——— of forced vortex centrifugal pump, 238
——— — propagation of a pulsation, 145
Ventilation, 19
Vertical short stroke engines, sizes of, 154
Volume and weight of saturated steam, 422
——— of certain solids, 1
Vortex chamber, 203, 208
—— ———, divergent, 212, 242
——, divergent free circular, 186
——, ——— —— spiral, 187
——, forced, 188, 212
——, free spiral, 184
——, — circular, 182

Warming buildings by hot water or steam, 16
Water wheels, design of, **1**
—— ——, Japanese, **27**
Weisbach, **19**

Weight and horse power of an engine, 429
——— — number of cylinders, 435
——— — volume of saturated steam, 422
Weights and probable errors of the unknown quantities, 405
——— of observations, 383
Wheel teeth, strength of, 38
Whirlpool chamber, 203, 208
Willans' engine, tests by Captain Sankey on, 425

宿屋の營業, **114**

材料强弱試驗報告, **58**, **59**
全國工業學校徒弟學校長會議要項, **108**
時間、時刻, **125**
實用數學摘要, **79**
上射水車, **9**
術語撰定の要旨, **60**

大正二年四月五日印刷
大正二年四月十日發行

不許複製

著作者　東京市小石川區西原町二丁目四十番地　井口在屋

編纂者兼發行者　東京市京橋區山城町十五番地　機械學會内　井口教授在職二十五年祝賀會

印刷所　東京市京橋區築地二丁目十七番地　株式會社東京築地活版製造所

印刷者　東京市京橋區築地三丁目十一番地　野村宗十郎

中に含んで居ります、可なり簡單なもので普通一般の本とは餘程趣が違つて居ります、此本は茲に置きますから御參考になりますれば御覽を願ひます先づ大體之位で終ります、尙ほ若し此件に就て疑問でもありましてお聞きになりたいと云ふ方がありますれば私の及びます丈はお答へ致します

ページ	著者名	書目
438	Saxelby	A Course in Practical Mathematics
407	Frank Castle	Elementary Practical Mathematics
382	Perry	Calculus for Engineers
406	Ormsby	Elementary Practical Mathematics
102 & 201	Briggs	Second Stage Mathematics

此本(Ormsby)は頁數凡三百八十あります、抑もの初めは算術であり、終りは微分積分迄になつて居ります、全部を二ケ年に教授すると云ふ考へである勿論抑もの算術から始つて二ケ年で遣ると云ふことは出來ないでありませう、先づ算術代數の初歩位迄進んだものを主として其等を遣り直すと云ふ考へでありませう、

此本(Perry)は同じく頁數凡三百八十、之は初めは代數から始つて居りましてさうして終りは微分方程式迄進んで居ります僅に之丈の大きさの本であります、

斯う云う小い本(Briggs)がありまず、之抔は程度の低い處にのみ用ゐると云ふ主意のものでありまして三角法位迄進んで居りまして幾何も其

た方が宜いと思ひます、何故ならば六桁の對數表では比例部分が大に過ぎて却て一層不便であります。

其から通常の算術書、代數初歩の本抔には時間、時刻に關係した問題を掲げてありますが、本文には時間、時刻に關係したことの說明を施さずして問題のみ宜く書いてある之は一般に不完全の點と考へる、時間時刻等に關係したことは力學の書に書いてありますが、併ながら算術の內にも重量の單位、時刻の單位其他色々の單位を教へて居る位であり、且つ時間時刻と云ふことは最も肝要な分量でありますから其に關した事柄を算術の本文に書いて置くべきである。尙ほ細部分に亘つて申上げたいですが、さう致しますと數學の講義のやうなことになりますからして大體の處は此位に止めて置きます、

尙ほ一つ玆に最後に申そへたいことは先刻申しました互に變化する分量二つを曲線で表はし之に依て變化の樣子を知り問題の解法を求めると云ふことは最も效力のある事柄でありまして之は皆さんも御存知でありませうが近年出版になりました本に多少書いて有ります、今玆に幾分携へて參りましたから御參考の爲め御紹介申して置きます。

の終りの桁に於ては多少の誤差は免がれぬと云ふことである、終りの桁に於ては單位の二つか三つ位は誤差があると云ふことは承知しなければならぬ、幾ら多く桁數を出しても用ゐた所の數が二桁或は三桁のものであり其末位に於て既に誤差を含むならば計算の結果に誤差は必らず生ずる筈である、故に三桁か四桁遣れば充分である、其以上は効能なし、効能がないのみならず、桁數を餘計にすれば間違いを爲す機會が多くなる、間違を爲す機會が多くなれば簡單なる數を扱つたよりも不利益である、尙ほ又二桁三桁ほか信用の出來ぬ數を用ゐて五桁六桁を出して居ると云ふことになると却つて人を詐くと云ふことになる、其であるから桁數を餘計に出すと云ふことは無駄骨折りであり、誤差を生ずる憂があり、尙ほ又人を詐くと云ふことになる、さう云ふ弊害を防ぐには計算尺を用ゐるのが、一番宜しい、人間の眼の力に限りがありますから其點に於ては計算尺と云ふものは重寶であります、計算尺では餘計な桁數まで進めることは出來ませぬ、併ながら若計算尺ではチツト粗雜過ぎると云ふ場合には對數表を用ゐる、四桁の對數表が最も宜しい、四桁に亞いでは五桁である、六桁の對數表は使はない方が宜い、四桁か五桁、其から飛んで七桁にし

めば正確であるか、四つも五つも計算すれば愈々正確であるかの如くに思ふ人が往々有りますが、正確不正確は小數點以下の桁數に依るものでは決してない何故ならば3.45貫(卽ち三貫四百五十匁)は3450匁(三千四百五十匁)と實際に於て勿論變りはない、前者には小數點以下二桁あるが後者は小數點以下何もない、眞價に於ては全く同じものであり數の粗密の度合も同じであり、唯其位置が相違して居るのみである、丁度一人物が二階に居るときと階下に居るときとの相違と殆ど同じ位であります總ての分量は多くの場合に於て名前を持つて居る、名前を持つて居る所の數は適當な單位を任意に撰定すれば皆小數點以上に登ほすことが出來る。金屬一立方の重量、材料の膨脹係數、比重强さ等色々な數が澤山ある、其數を見ますと桁數三つ位より多いのは稀れである、大抵は二桁位で止つて居る、尤も空氣、水、水銀、などは物理學、工學には肝要なる標準物體でありますから三桁或は四桁位も出してある、水は他の物體の物指であるかの如く取扱はるゝ物品であるから其性質を表はす數は桁數多く特に精密にして置かねばならぬ、併し工業家が日常取扱ふ所の材料の性質を表はすには二桁位で澤山である尤も此處に注意して置かねばならぬことは數

者を各々三倍する云云と說明する。又比例と云ふものは米の値段は米の値段、桝數は桝數と相對しなければならぬと云ひますがこれも愚な話である、一つの數を同種類の數で除さなければ比にならぬと云ひますがなるならぬは問題の答解には關係せぬ。等しい分量を二ツ求めて其れを等號で結び附ける、そうしてただわかりを應用する、斯樣にいたせば何時の間に代數に這入つたか分からぬように代數に這入つて仕舞う、さうすると代數と云ふものは容易いものであると云つて喜んで遣るやうになるさう云ふ方法で遣るが宜しい。

夫から、順序がちと前後しましたが、大さの事を云ひ落しました、大さは斯うです、小い方は前述べたやうに寸法で云ふならば物指の二厘目位迄は眼で分る、其分ると云ふことが土臺にならねばならぬ、大い方では地球の大さ、或は天文學に於ては光が地球に達する爲めに何十年を要すると云ふ莫大な距離、是等は工業家には必要でありませぬから止めて置きます、先づ大さはどれ位迄と云ふ程度を考へて置かなければならぬが總て數量は其れの大小に關はらず桁數四つ位あれば充分であると云ふことを承知して居つたら宜しい、數量を計算するに當て小數點以下幾桁迄進

學に於て初めて述べ立てるが普通である、併し抑々公理とは殆んど說明なくして容易に理會できる道理である、卽ち馬鹿でない以上はわかる筈の通理である、卽ちただわかりである、ただわかる位な道理は小學校生に對して敎へても何等の不都合はない、今大きい分量と小さい分量とがある、之れに同じ分量を各々加へる、すれば元大きかつたのは今も矢張り大きい、斯樣な事は勿論ただわかりである、因て此ただわかりを利用して等號の運用を敎へる、例へば今此處に米一升の値段が二十錢とする、一升の値段二十錢三升の値段幾らと云つたならば三升の三を二十錢に乘じさへすれば宜しいが、始に一升と云はずして四升の値段一圓二十錢、三升の値段幾らと云ひますならば普通には比例式

$$4 : 3 :: 120 : x$$

を書き稍無意味的に二項三項の積を一項で除る斯くするよりも

$$\frac{120}{4} = \frac{x}{3}$$

と書き、左右各々一升の値段であるから相等しいと申して置く、此等しい

難いものである、殊に分數に關係した最小公倍數とか、最大公約數と云ふ事柄は之は畢竟算術の內に置いたのは無理でありませう、もうチツト代數的頭腦が出來まして其上であれを遣りますと宜く分ると思ふ、算術に之を置くのは或は已むを得ぬか知らぬが繩梯子的であるかと私は思ひます、三分の一とか四分の一とか云ふ簡單な分數を實物にでも就て教へる、さうして直に小數に移ると云ふ方が宜く分ると思ふ、最小公倍數、最大公約數の如きは代數に於て之を學べば僅な時間を以て容易く分る、一體算術と云ふものは研究すればする程深みにはまり込み次第に面白味が出で來て益々實用に遠ざかります、現今の遣方は丁度子供が道に迷つて居るやうなものだと思ひますが、算術は成るべく早く切揚げて仕舞つて二階三階に登ることを工風しなければならぬと思ふ、二階三階に登るに就ては便利な梯子の段を取り設ける。其算術の範圍內に於ても等號卽ち左右兩式の値相等しと云ふことを表はす＝符號の使用を教へる、此符は小學校程度の生徒にも運用は容易に出來る、嘗て若干の生徒に對して試みた結果頗る有望であつたとの事を聞き及んで居ります、此符號を用ゐるに附ては所謂公理と稱する者を教へて置かねばならぬ、公理は幾何

容易に出來ます、是れが卽ち計算尺の使用である、しかるが故に有らゆる數量の精粗の範圍は計算尺の精粗の範圍で代表することが出來る、今長さ十吋の計算尺、虫眼鏡の附いて居らぬものを使用して任意數量を計算し、使用者の眼力を六十四分の一吋と假定する、然るときは計算の結果は全量の約六百四十分の一までは確である、是れ卽ち精粗の範圍である、計算尺を使用すれば目盛と眼力とに制限せられて右の範圍忘却は自然に出來なくなる、是れが計算尺の効能の一つである。普通は長さ十吋の計算尺が廣く行はれて居る、或は特別な虫眼鏡を備へ附け長さ五吋のもので十吋代用となるものも有る、教授用としては二十吋位なるものを備えて置く必要がある先生が之を持つて居つて生徒の計算したものを見て檢べて遣らねばならぬ、先生の方では生徒の方で計算した結果を檢べて遣らねばならぬから、稍々精密なものを使ふ、十吋の代りに二十吋卽ち倍位迄分るものを持つて生徒の誤りを正して遣る必要があると思ひます。

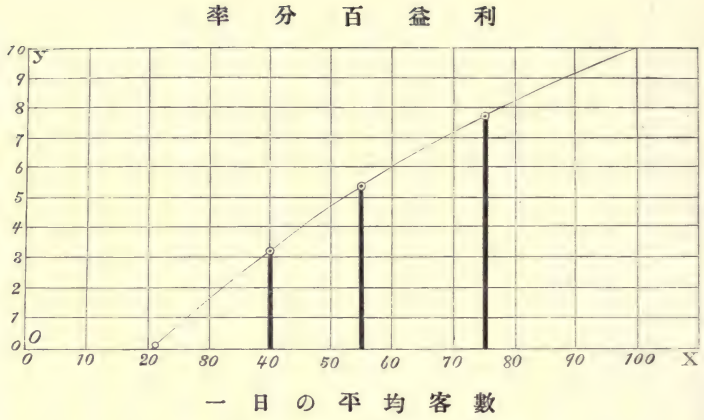
夫からもう少し今度は細かい處に亘りまして、卽ち數學全般の教方の順序に就て細かい處を云つて見ます、普通は算術、代數、幾何と云ふやうな風に進みますが其算術の範圍内に於て分數と云ふものは結局大に分り

考へねばならぬ、一體數は右申した樣に增減するものであるが如何なる程度まで果して增進し又は減退し得るものか、卽ちどの位大きい處迄行けるか、又どの位い小い處迄行けるかと云ふことが問題である、人間の視覺と云ふものは甚だ微弱なものであつて極く大な數にも考へが付かず、又極く小い數にも考へが付かぬ、故に先つ人間の眼の力に依つて直接に見得る丈が寸法と云ふ數に對する普通範圍である、例へば物指で云へば、分迄は老人の極く眼の惡い人にも分る、二厘位ならば眼の力のよい人には容易に分る、一厘となると一寸見難いでせう、であるから一厘と云ふ目盛は付けて置かない方が宜い、兎に角肉眼で見え得る程度、物指ならば二厘位迄は印を付けても宜い、尙ほ夫より細かい寸法を測り讀む方法は普通物指及び肉眼以外に澤山あります、緻密なる割で吾々の取扱ふものは一インチの百萬分の一迄調べることが出來ます、夫は一種特別な遣方で專門家の遣ることであります、然るに普通の物指の目盛を肉眼で讀んで得た所の結果を使用して厘毛以下までを計算する如きは數の精粗の範圍を忘却したと申すものである此種類の忘却が往往あります。さて物指で測り得る所の寸法を用ゐて有らゆる數量の代理を勤めさせる事が

云ふ單位を使ひ、其次に單位秒と云ふのを使ふ、皆個々別々のものゝやうに思ひますが時の進行するのは、時で一つ〳〵、分で一つ〳〵、秒で一つ〳〵進むのではない漸を追ふて進むものである、又大氣の溫度も其通りで昨日は暖で八十度であつたが今日は寒くなつて五十度だと云つても八十度から瞬間に五十度になつたものではない、總ての點を通過せねば次の溫度になるものではない、さう云ふ分量を數で表はして居ると云ふ譯でありますから、數と云ふものは漸々と進むものであつて一つ〳〵一單位づゝ急に飛ぶものではないと云ふ觀念を第一着に付けねばならぬ、さうして見ると普通の數を勘定するにも小數點を付けて其數位を示し其次に又數字を置くと云ふ風にして數の觀念を付けねばならぬ譯である、通常吾々が使ふ所の數は殆と皆夫である、米とか、麥とかと云ふものを、一升二升と勘定するが其升は品物の區切でなくして分量の目安である、渡すにも受取にも升がいくつ合がいくつと云ふ區切りは決して目には見えない、重量でも値段でも皆漸を以て進み、漸を以て退くべきもので總て皆な極く少部分から成立つものである、と云ふ考へを付ける必要があると思ふ、偖てさう云ふ觀念を着けるに就ては數量の大小精粗の範圍を略ぼ

題は普通一般の數學の教科書等には書いて有りませぬが他には殆んど方法がない譯であります、斯う云ふ風にして解きますと數學的の頭のない人にも容易に分り、さうして手數も掛らず重寶である、

夫から今一つ數量の說明と云ふ事を申しませう、一般に數と云ふ觀念を普通教育程度ではチツト輕く視過ぎて居るかと思ふ、一般に數と云ふものを子供の時分から一つ二つ三つと云つて勘定させて單位宛增加するものと云ふ觀念を專ら與へる、無論始めはさうであつたらうと考へる併し、今此處に團子があるとする、二つ三つ四つと勘定する、其一つを半分に割つたらば其各は小數點以下五と云ふ考へを附けねばならぬ、多少說明すれば夫位の考は次第に附いて來る、又ものを三つに等分して三分の一と云ひますがこれは又同時に小數點以下三三であると云ふ考へを附けさせて置く方が利益である、抑々廣い意味の數と云ふものは個々別別な單位づゝ增減するものでなくして、いつとなく、いくつとなく、句切りもなく、段もなく、漸を以て、勾配を以て、離れず、切れず、減つたり、殖えたりする分量を遺憾なく表はし得る者である、例へば時の進行の程度を表はすに何時何分何秒と云ふ數を用ゐる、此場合には時と云ふ單位を使ひ、分と



百五十ミリと見做して印をつける、又垂直線OYを引き四十人に對する利益の割合百分の三、三を三十三ミリとして四十人の印の處からOYに並行に引き點をしるす、又五十五人に對する利益の割合百分の五、四三を五十四、三ミリとして五十五人の印の處からOYに並行に引き點をしるす、又七十五人に對する利益の割合百分の七、八を七十八ミリとして七十五人の印の處からOYに並行に引き點をしるす、右の三點を滑かな曲線で結び附けて之を三點以外まで多少引き延ばす、然るときは平均客數八十五人に對する利益は凡そ八分七厘位百人に對する利益は凡そ一割であることが圖面で容易にわかる、尙ほ又利益なし損失なしと云ふ場合は平均客數凡そ二十人のときに相當する、斯う云ふやうな問

今或處に宿屋の營業を開いた、設備は夜具百人前、膳部も百人前、お臺所の料理道具も大凡百人前に適當するものを拵へて營業した處がお客は日日四十人位、隨て利益は勿論最初に豫算したよりも少い、併し全く損耗でもなく、百分の三か四位な利益はある、宿屋の主人は之ではいけぬと云ふので大に骨を折つて客人の御意にかなふ樣になり日々十五人程殖えた、今度算當して見ると利益が百分の五六位になつた、其から又さうこうして居る內に更に殖えて日々七十五人計りになつた、夫で計算して見ると利益の割合が餘程殖えて居る、其處で四十人、五十五人、七十五人と云ふ員數に對して利益の割合が各々出て居る、さて員數八十五人のときには利益は幾何か、又當初の豫定通り、若し百人位宛日々お客が有たならば利益が果して幾ら位になるだらうかと云ふ問題、之は算術でも出來ず、代數でも出來ず困つて仕舞う、併し之は實際問題であり大略の解法は必らず出來る筈である、して其方法は頗る容易である、紙を一枚取り水平線を引いて物指を用ゐて距離を示し之を人員と假定する、夫れから垂直線に距離を取り之を利益の割合と假定する、尙ほ畧圖に就て申します、水平線OXを引き之に沿ひ人員四十人を八十ミリ、五十五人を百十ミリ、七十五人を

看做して時間を水平寸法で表はし、距離をば垂直寸法で表はし、其距離と時間との關係を點で表はし、是等の諸點を通過する或は諸點に最も接近したる滑かな曲線を引く、此曲線に接觸した直線を引く、此線の勾配が卽ち時間に對する距離の微分係數である、卽ち速さである、右は極く容易い方法である、汽車旅行中時計を持つて居れば汽車の線路の側には距離の標柱が立つて居るから、其標柱と時計とを合せて手帳に記入する、家に歸つてから紙の上に前の方法で曲線を引く、そうして任意接觸點に於て角度を計つて其正切を求め之れに距離と時間との縮尺割合を掛ける、是れ卽ち速さである、斯て微分の根本たる處の考へが容易く得られる譯である、さう云ふ方法を講じますと云ふと餘程程度の高い處迄も進めることが出來る譯である、

其からもう一つ種類の變つたことを申します、通常算術の本抔には問題を解くときには乘ずるとか、除るとか、減くとか、比例式を立てるとか、平方根に開くとか、何とか法式があつて夫に依つて結果を出すと云ふことになつて居りますが、或る場合には乘じても結果は出ず、除しても減いても平方根に開いても出ない、斯やうな問題が普通實際に隨分ある、假令ば

すると云ひますが併ながら目的は目的の處があるのでありますから、其方に進むことを考へねばならぬ、二階に登るならば勾配の急でない段を作ると云ふ方法を案出しなければならぬと思ふ、

其處でもう一つ肝要なる點は數學の內にも色々な分科がありますが、其分科の內で最も効力のあるのは微分積分です、微積分を多少理會することが出來れば其利用も大に出來る譯である、併ながら普通の程度の處では微分積分抔を教へることは工合が悪いが、微分積分と云ふ考を與へるのはさう六ケ敷いことではない、今汽車が停車場から發して程遠からぬ間は其進行が徐いと言ひます、停車場を出て平坦な線路を進行すれば暫時の後に大層疾やく行くと人が言ひます、此徐い疾やいは田舍の老嫗さんにもよく解ります、してその徐いこと疾やいことは微分係數の大小です、されば田舍の老嫗さんにも微分係數はよく解ります、さて明瞭の意味を以て數の上で此等の微分積分の根本の考へは得られるのであります、其根本の考へを得させるにも普通程度なる小學中學程度でも六ケ敷くはない、今汽車の速さの事を申しましたが、汽車の速さは距離と時間との關係である、長距離進行に於て其時間と距離とを互に變化する分量と

點あるかなしである、四十點か五十點位であつても宜いと云ふ風にして進む、一方に於ては其中には十分の一位は秀才が居る、一割位か或は百分の五位かも知らぬが、其秀才を進めると云ふことが肝要である、尻尾を進めるよりも百分の五の方を進める方が國家のためには却て利益である、全體の平均は進めて行かねばならぬが、尻尾の四分の一位は捨てる積りで頭の方の百分の五は骨を折つて進めて行くと云ふ風にして行きまして全體一緒に平坦な教授をするよりも餘程工合が宜くなる譯であります、で學級學校を卒業しました後に至つて教へられた生徒全部が數學的の事許り應用する職業に就くならば全部同じようにしなければならぬが、或る部分は同じ工業に從事しても事務の方に關係するとか、工場の現業に關係するとか云ふ風になる譯でありますから必ずしも全部數學的頭腦を作らねばならぬと云ふ譯はない、且つ本來人間の頭腦には限りがあるから數學に向かない頭腦に強いて數學を入れやうとするのはチツト無理である、向いた方の頭腦をずん〳〵進める方が宜しいと私は考へる、さうして一方には教へるには繩梯子で二階に登ると云ふ方法は採らない、六ケ敷い問題は出來る丈除けて仕舞う、さうして簡單にする、簡單に

其道具をどれなりとも使つて鑿でも鉋でも錐でも何でも使つて考へて進歩すべき筈である、然るに一々之は代數である、幾何である、此問題は幾何で解くべきものであると云ふことは抑も間違つて居る、立派な道具が澤山あるのであるから其内のどれでも使はせねばならぬと思ふ、數學を教へるには之に髣髴たる順序に依つて教へたならば、工業進歩の主旨に最もよく適ふではないか、結局目的に達することが早いと考へる、

其から今一つは數學的の事柄、卽ち物理抔にしても數學の應用が澤山ある譯であるが、其等の學科は幾分か他の文學的學科或は業務的學科とは種類が幾分か違いまして人々の頭腦に依つて中には幾ら勉強しても進むべき處迄進むと云ふことの保證が出來ない人物があると思ふ、であるから今一つの學校に三十人なり、五十人なりの生徒がありまして之に數學を教授すると云ふ場合を考へます、此全體の級を進めるには平均で進むと云ふことが肝腎でありまして平均が進まない以上は無闇に進めると云ふことは出來ない、全體の平均を考へて其に依て進むべきである、併し數學に於ては末席の方に居る處の四分の一とか、五分の一位は數學のような學科に就ては屑であらうと思ふ、試驗の時に人員二割位は五十

にあるやうに思ふ、近頃は如何か知らぬが、一寸例を申しますと、以前は海軍省の兵學生を募る時に試驗問題を拵へて一般から生徒を募ると云ふ工合で數學、英語等の試驗をした、其試驗問題を見ますと、普通の算術で解すべき問題が澤山ある、其問題たるや如何にも六ケ敷い、例へて云ふと今二階に登らうと云ふ時に梯子の段を備へず繩を一本與へて其先に鉤を着けて其鉤を引掛けて二階に登れと云ふやうな問題が澤山ある、低い處から二階に上るにはチヤンと立派な適當な段梯子がありながら繩梯子で上れよ、斯う云ふやうなことを遣つたものであります、上ることが味く出來れば巧妙な働きがあるものだと思つて居るが之は間違いである、段梯子が有るならば其道を探して二階なり三階なりへずん〳〵速に登るべきであらうと思ふ、

普通の數學の順序を言ひますと先づ算術を教へる夫から代數、幾何、三角法と云ふ風な順序で進む、此順序も後に作つた順序で數學が發達した時にはさう云ふ順序で發達する筈はない、算術で考へる幾何で考へる、三角法で考へる代數で考へる、さう云ふ變な科目を分けてから頭腦の中で考へると云ふことは無論ない、數學と云ふ立派な道具があることなれば

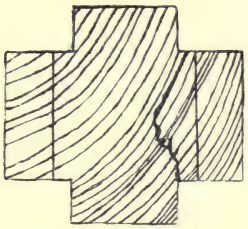
十一。

工業的數學。

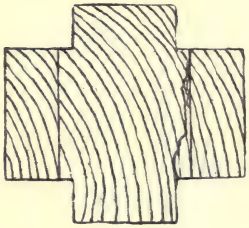
〔全國工業學校徒弟學校長會議に於て講演。文部省實業學務局編、全國工業學校徒弟學校長會議要項、所載、明治四十四年十二月。(1911)〕

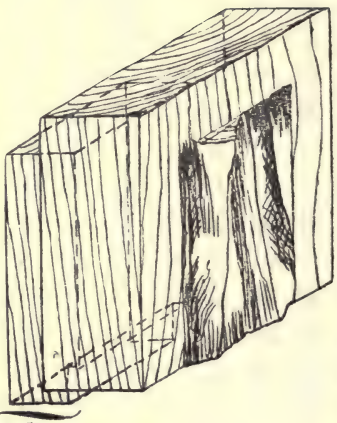
眞野局長の御依頼に依りまして一席工業的數學の事をお話する樣にと云ふので、出まして、何か申上げることになりました、私は豫て幾分か工業に關した數學めいた事柄を心掛けて居りますに依つて幾分かお話の材料になることもありますが併ながら、其全部が果して數學者、工學者の是認して居る事柄であり正しい論法であるかないかは或は保證出來ないかも知れぬ、其要項を申上げて諸君に於て其取捨は御自由になされんことを希望致します、或は申上げることが極端に行過ぎる事がないとも限りませぬで、夫等の處は然るべく取捨を願いたい、數學の教授法に就て豫てより考へて居ります、第一の事項は一般に進むことの遲い方法を用ゐて居る、卽ち餘り順序正しく遣り過ぎて却つて進まないと云ふ弊が大

第六圖の五



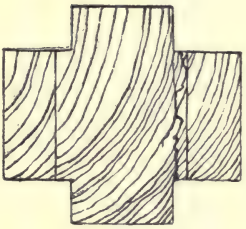
欅
12の1



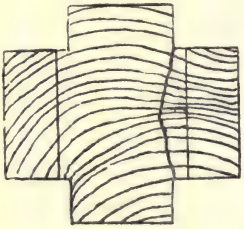


朴　2の1

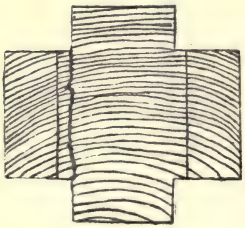
楠　23の3



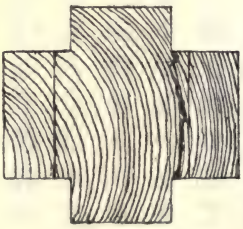
櫻　4の1

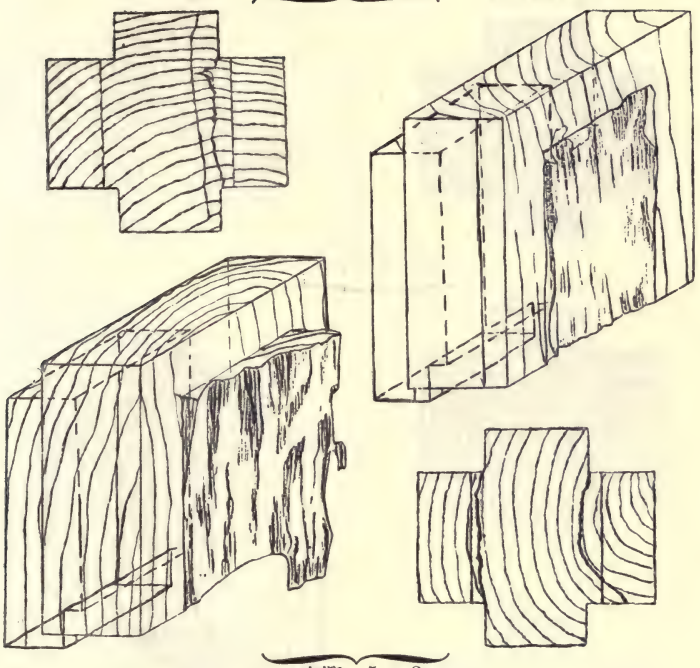


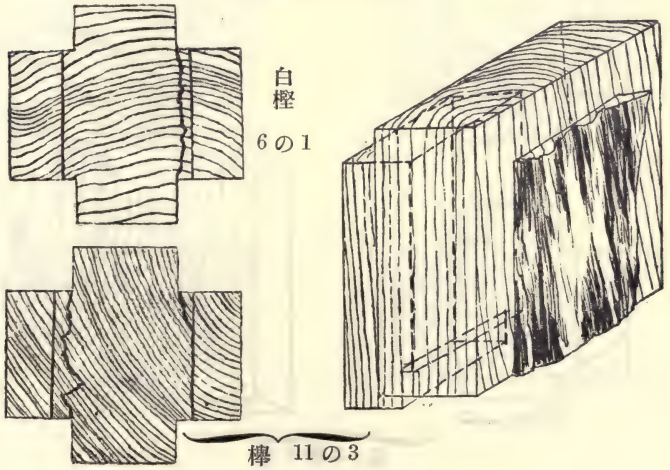
刺桐　22の4



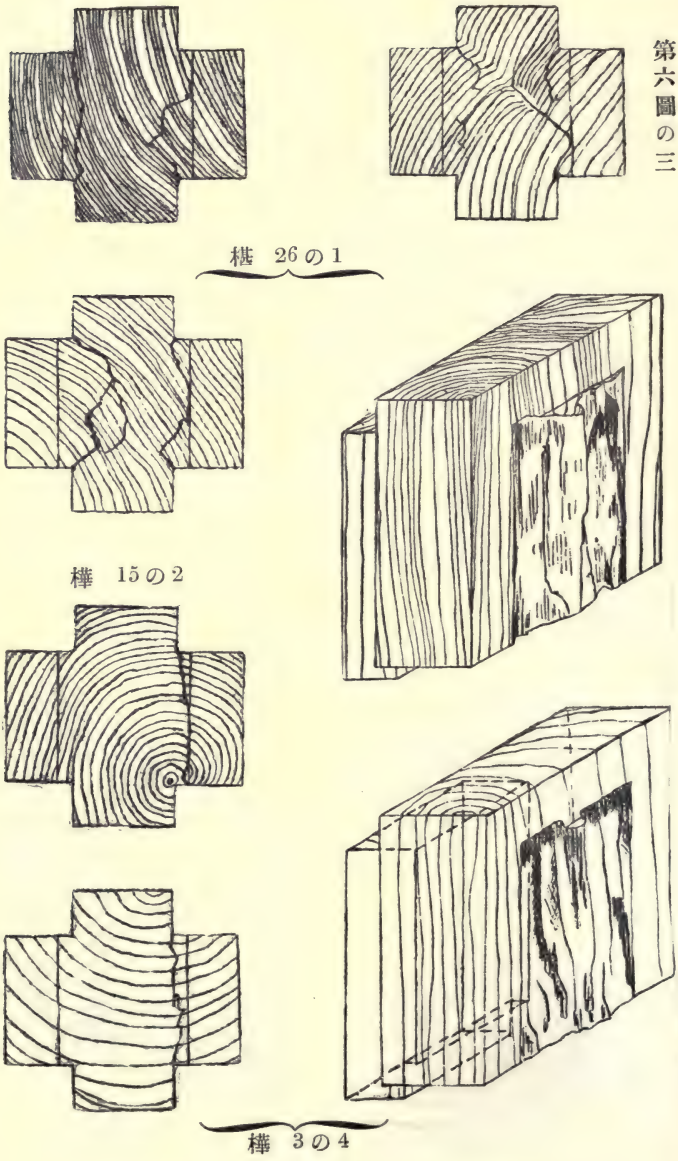
鹽地　20の2




赤樫　5の1
赤樫　5の3

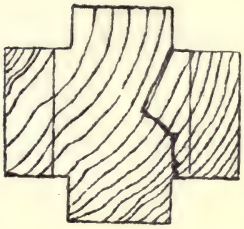

白樫
6の1
欅　11の3

第六圖の三

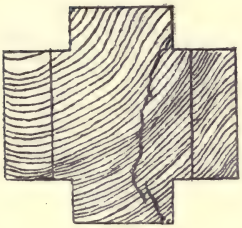


第六圖の二

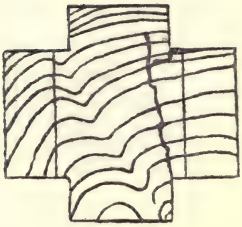
杉　9の1



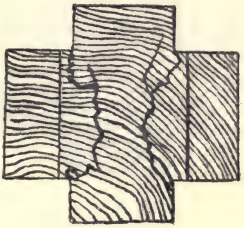
杉　10の2



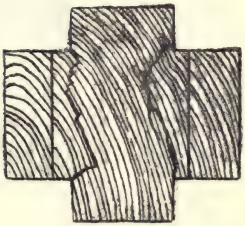
杉　27の3



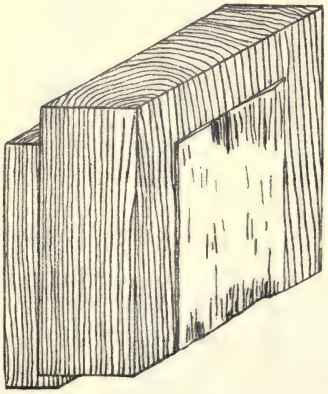
杉　28の4



檜　30

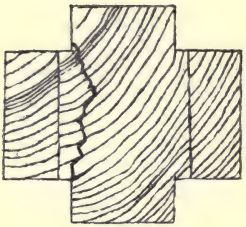


檜　32

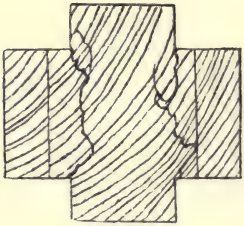


第六圖の一

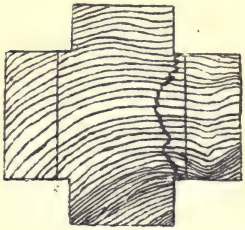
松　14の2



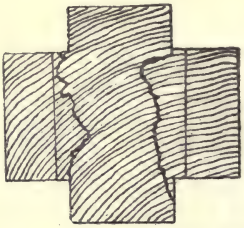
松　18の2

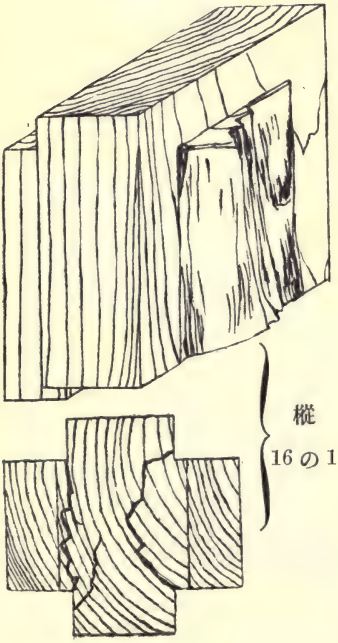


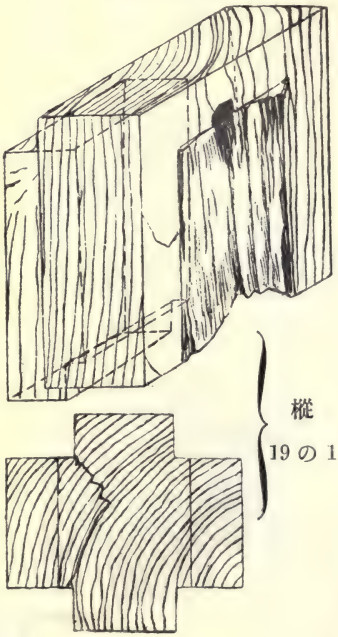
栂　8の4



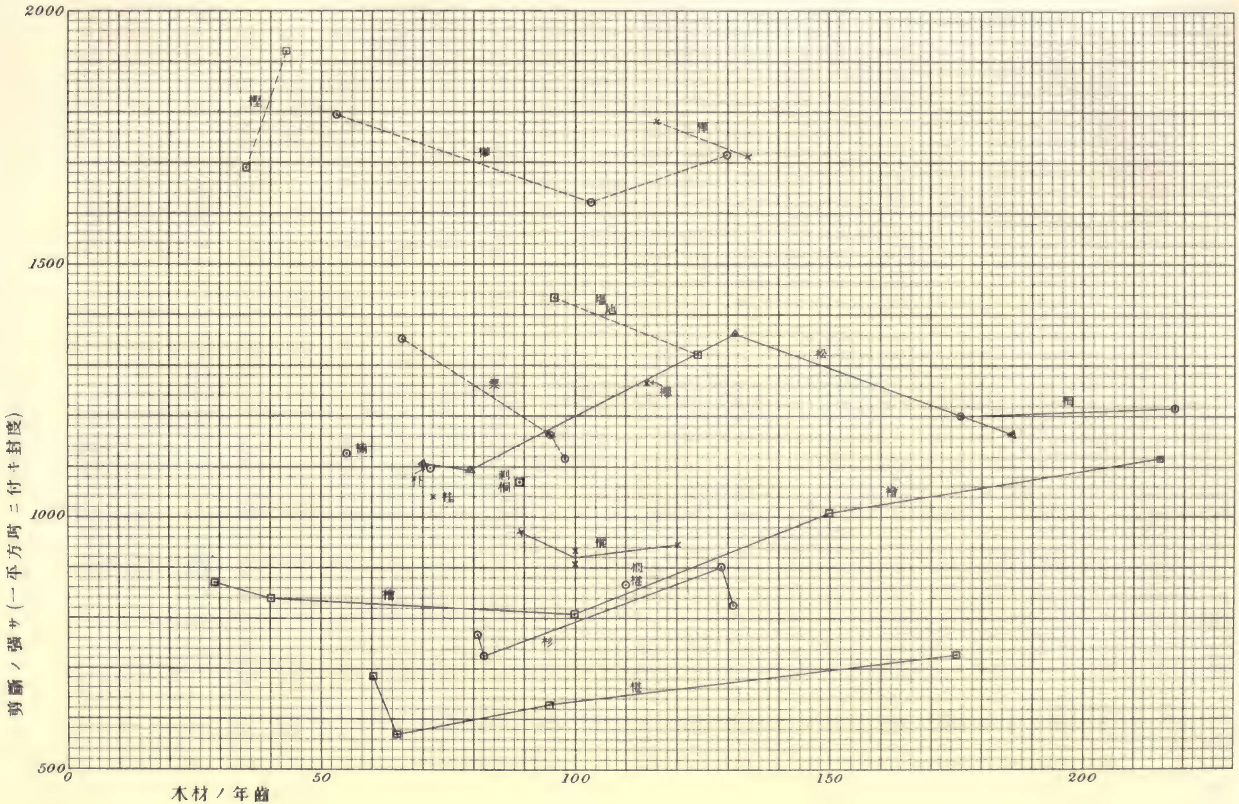
栂　17の3







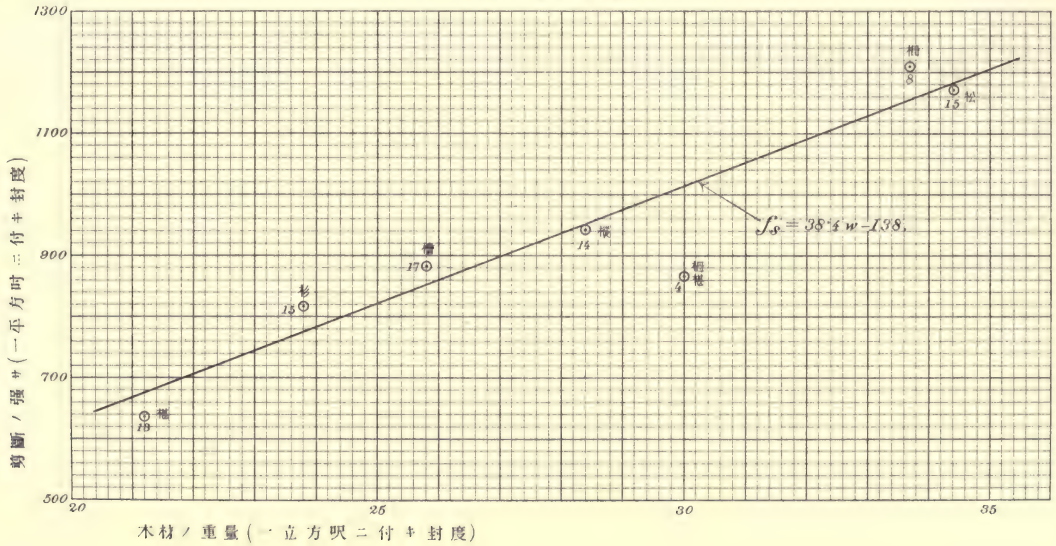
第四圖



第參圖

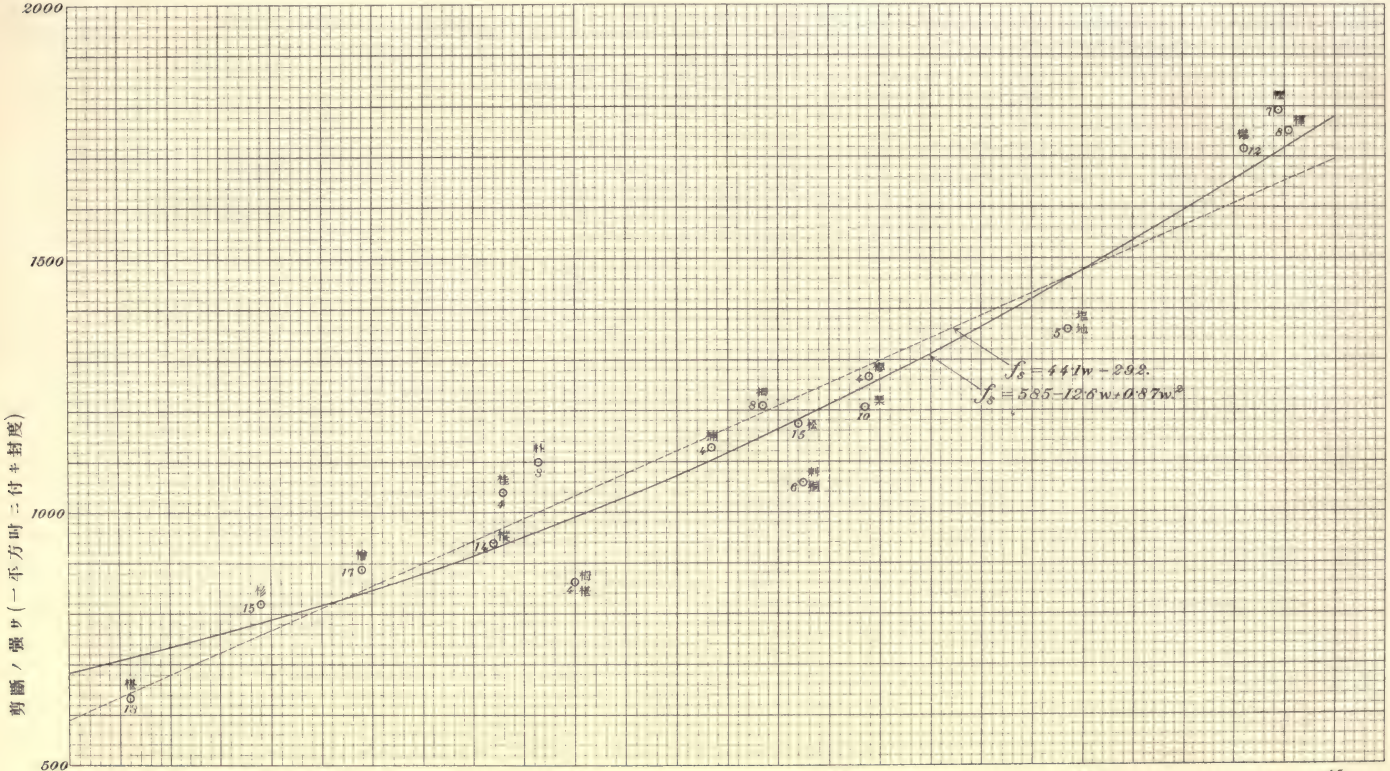
点ノ側ニ在ル數字ハ試驗片ノ數ヲ示ス

(松杉科木材)



第貳圖

点ノ側ニ在ル數字ハ試驗片ノ數ヲ示ス



となり、試驗片に於ける破壞總荷物より直ちに算出するを得べし

第六圖は破壞後の試驗片の底面及び破壞面の狀況を示す、之を見るに破壞は必ず年輪に沿ひて起る、然らざれば之に直角なる方向に起る、未だ其他の方向に於て破壞せしものを得ず。

破壞面上には微細なる粉末狀をなせる木屑の附著するを見る、特に欅に於て著し。

（明治三十九年三月）

最大曲げモーメントは梁間の中央に於て起り、其値は

$$M_{a+\frac{1}{2}l}=\frac{1}{4}W(a+l-b).$$

第一圖に示す所の梁の寸法を此式に挿入して計算すれば、梁間の中央に於ける最大曲げモーメントは

$$M_{max.}=0{\cdot}484W.$$

而して梁の強さに對する通常の公式は

$$M=f_b\frac{I}{y}$$

なるを以て、之より曲げ内力の値を見出せば

$$f_b=\frac{yM}{I}=\frac{6W}{bd^2}$$

$$=\frac{6\times{\cdot}484\times W}{35\times4^2}$$

$$=0{\cdot}052W$$

力を兩足の下面に受け、梁間の一部分に平等なる荷物を受けたる梁と看做すを至當とす、而して反動力も平等に分布せるものと假定すれば任意橫斷面に於ける曲げモーメントは次の如くなるべし。

第五圖に於て、A B 間に於ける曲げモーメントは

$$M_x=\int_0^x\frac{W}{2a}XdX=\frac{1}{4a}Wx^2$$

B C 間に於ける曲げモーメントは

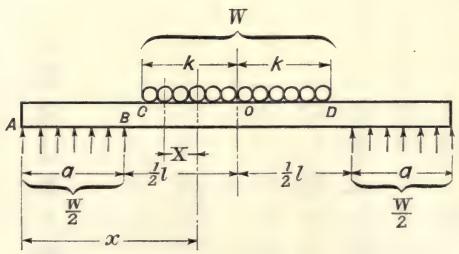
$$M_x=\int_{x-a}^{x}\frac{W}{2a}XdX=\frac{1}{4}W(2x-a)$$

C D 間に於ける曲げモーメントは

$$M_x=\int_{x-a}^{x}\frac{W}{2a}XdX-\int_0^{x-(a+\frac{1}{2}l-k)}\frac{W}{2k}XdX$$

$$=\frac{1}{4}W(2x-a)-\frac{1}{4k}W\left\{x-(a+\frac{1}{2}l-k)\right\}^2$$

$$=\frac{1}{4}W\left\{(a+l-k)-\frac{1}{k}(a+\frac{1}{2}l-x)^2\right\}.$$



第　五　圖

$f_s = 44{\cdot}1w - 292$ ……(2)

を以て其大凡を表はし得べし、第二圖に點線にて示す所の直線は卽ち是なり。

今、松杉科の木材のみに就きて稽ふるときは w と f_s との關係は第三圖に示す如く

$f_s = 38{\cdot}4w - 138$ （松杉科木材）

なる直線式を以て表はし得べし。

木材の年齒と其剪斷强さとの關係を求めんが爲め、此二者をコルヂネートとして點を配記したるに第四圖の如き分布となり其間に確たる關係を見出し難けれとも松、檜、椹及び杉の如き松杉科木材は年齒八十年より二百年に涉る間に於て其剪斷の强さは年齒に伴ひ多少增加するが如し又欅及び栗の如き比較的重き木材に在つては年齒百年前後に於て其剪斷の强さは多少減退する傾向あるが如し。然れども此事實に關しては本試驗の結果により充分論斷するを得ず。

曲げ作用によりて破壞する場合を考ふるに、各試驗片は分布せる反動

とをコルヂネートとして點を配記したるに木材の種類に係らず何れの點も狹き彎曲帶内に在るを認めたり、因て此關係を表はす所の式

$$f_s = A + Bw + Cw^2$$

式中　f_s 封度は一平方吋に付き剪斷の強さ、
w 封度は木材一立方呎の重量、
A、B、C は定數、

を假定し、第二圖に於て諸點を平均する曲線を求めて

$$f_s = 585 - 12{\cdot}6w + 0{\cdot}87w^2 \quad \cdots\cdots\cdots\cdots (1)$$

を得たり、圖中に實線にて示す所の曲線は卽ち是なり、而して此關係は本試驗の範圍内卽ち木材一立方呎の重量が二十封度より四十五封度の間に適用せらるるものなり。

各種の木材に就きて別別に稽ふるときは w と f_s との關係は概略直線式と看做すことを得べし、然れども各種に對する實驗の數未だ多からずして式を作るには充分ならざらん。又本試驗の範圍内に於て諸種の木材に共通なる公式(1)は簡易なる直線式

らんが爲め、第一表に就き種々研究したるに、試驗數少くして充分にはあらざれども、二樣の計算法を用ゐて次の結果を得たり。

第一の法は、本試驗に於ては曲げ作用によりて破壞したる場合の破壞荷物は剪斷によりて破壞すべき破壞荷物と看做すとも大差なしと假定し、同一原材の內に於て年輪に沿ひての剪斷強さと之を橫切りての剪斷強さとの平均を求め其割合を算出して第三表を得たり。之に據れば年輪に沿ひての強さは之を橫切りての強さの97.5%に當るを見る。然るに此第三表中に揭ぐる如く、或種の木材に於ては其試驗數は單に一個のみなれば之によりて算出したる割合は平均に近からざる虞あり。因て茲に第二の法を試む、此計算法に於ては同一原材の內にて完全なる剪斷作用によりて破壞せしもののみを撰出し、且つ二個以上の平均數を得らるるもののみに就きて關係を見出すに在り。其結果は第四表に示すが如し。之に據れば年輪に沿ひての剪斷強さは年輪を橫切りての剪斷強さの92.6%となる。

木材の強さと其比重とは一定の關係を有すること一般なる事實なるが如し、此關係を索むる爲め第二圖に於て剪斷の強さと一立方呎の重量

第三表

材名	試驗片の數		年輪に沿ひての剪斷強さ／年輪を橫切りての剪斷強さ %
	年輪を橫切りての	年輪に沿ひての	
松	4	5	80.4
栂	6	2	91.3
樅	6	4	92.8
杉	8	4	100.0
檜	7	4	112.0
椹	4	4	97.5
欅	5	3	101.2
栗	3	2	87.4
赤櫧	3	1	101.7
白櫧	1	2	86.8
欅	5	2	89.8
朴	2	1	105.5
楠	2	2	98.4
櫻	3	1	127.0
鹽地	3	1	90.4
平均			97.5

第四表

材名	試驗片の數		年輪に沿ひての剪斷強さ／年輪を橫切りての剪斷強さ %
	年輪を橫切りての	年輪に沿ひての	
松	3	2	77.0
杉	3	3	102.8
欅	4	3	95.5
栗	2	2	92.0
櫟	5	2	89.8
楠	2	2	98.4
平均			92.6

第一表の六

材名	番號	產地	年齒	重量 一立方呎に付き封度	荷物の掛け方 年輪に，を	破壞總荷物 封度	剪斷の平均結局强さ 一平方吋に付き封度	破壞の狀況
鹽地	20	秩父	124	40.8	橫切りて	13,760	1,310	剪斷
				40.3	沿ひて	13,850	1,224	剪斷
				39.2	橫切りて	14,960	1,425	曲げ
				38.7	橫切りて	13,990	1,332	曲げ
平均				39.8			1,323	
鹽地	21	秩父	96	39.6	橫切りて	15,080	1,436	曲げ
鹽地總平均				39.7			1,355	

第二表

材名	重量 一立方呎に付き封度	剪斷の平均結局强さ 一平方吋に付き封度 最小	最大	平均	試驗片の數
松	34.4	961	1,520	1,170	15
栂榧	30.0	831	913	862	4
栂	33.7	1,107	1,290	1,206	8
樅	28.4	820	1,020	942	14
杉	23.8	695	950	815	15
檜	25.8	650	1,350	882	17
椹	21.2	542	733	635	13
樺	44.1	1,510	2,040	1,749	8
栗	35.7	1,030	1,381	1,206	10
赤櫻	42.7	1,627	1,721	1,692	4
白櫻	45.6	1,730	2,100	1,917	3
欅	43.2	1,450	1,990	1,712	12
朴	29.3	1,055	1,156	1,100	3
桂	28.6	1,003	1,061	1,039	4
楠	32.7	1,034	1,205	1,129	4
櫻	35.8	1,078	1,501	1,263	4
刺桐	34.5	1,004	1,110	1,057	6
鹽地	39.7	1,224	1,436	1,355	5

第一表の五

材名	番號	産地	年齒	重量 一立方呎に付き封度	荷物の掛け方 年輪に、な	破壊總荷物 封度	剪斷の平均結局強さ 一平方吋に付き封度	破壊の狀況
欅	11	遠江	130	42.8	横切りて	18,090	1,723	剪斷
				42.3	横切りて	18,150	1,729	剪斷
				39.0	横切りて	18,100	1,724	剪斷
				44.1	横切りて	17,800	1,695	曲げ
	平均			42.0			1,718	
欅	12	遠江	103	40.6	沿ひて	15,200	1,450	剪斷
				41.7	横切りて	17,770	1,695	剪斷
				41.3	横切りて	17,690	1,685	剪斷
				40.7	横切りて	17,480	1,665	剪斷
	平均			41.1			1,624	
欅	37	地木	53	43.8	横切りて	17,400	1,700	剪斷
				——	横切りて	20,250	1,990	剪斷
				48.8	沿ひて	17,200	1,690	剪斷
				49.7	横沿	18,300	1,800	剪斷
	平均			47.4			1,795	
欅	總平均			43.2			1,712	
朴	2	北海道	71	29.7	沿ひて	12,130	1,156	剪斷
				32.5	横切りて	11,080	1,055	剪斷
				25.7	横切りて	11,355	1,090	曲げ
	平均			29.3			1,100	
桂	13	北海道	72	29.3	横切りて	10,890	1,037	剪斷
				28.3	横切りて	11,090	1,054	剪斷
				27.7	横切りて	10,530	1,003	剪斷
				29.1	横切りて	11,140	1,061	剪斷
	平均			28.6			1,039	
楠	23	伊豆	55	31.8	沿ひて	10,860	1,034	剪斷
				36.3	横切りて	12,000	1,143	剪斷
				31.2	横切りて	11,930	1,136	剪斷
				31.5	沿ひて	12,650	1,205	剪斷
	平均			32.7			1,129	
櫻	4	新宮	114	35.6	横切りて	11,320	1,078	剪斷
				35.9	横切りて	12,000	1,143	剪斷
				37.5	沿ひて	15,760	1,501	剪斷
				34.4	横切りて	13,950	1,329	剪斷曲げ
	平均			35.8			1,263	
刺桐	22	北海道	89	35.1	横切りて	11,650	1,110	剪斷
				34.6	横切りて	11,320	1,078	曲げ
				33.8	横切りて	11,290	1,075	剪斷
				33.7	横切りて	10,540	1,004	剪斷
	平均			34.3			1,067	
刺桐				——	横切りて	10,870	1,035	剪斷
				35.2	横切りて	10,800	1,030	剪斷
	平均			35.2			1,032	
刺桐	總平均			34.5			1,057	

第一表の四

材名	番號	産地	年齒	重量 一立方呎に付き封度	荷物の掛け方 年輪に、を	破壞總荷物 封度	剪斷の平均結局濶さ 一平方寸に付き封度	破壞の狀況
樺	3	新宮	116	45.0	沿ひて	18,910	1,810	剪斷
				45.4	横切りて	19,280	1,836	剪斷
				45.0	沿ひて	18,670	1,778	剪斷
				45.2	横切りて	18,000	1,714	剪斷
	平均			45.1			1,784	
樺	15	新宮	134	45.2	沿ひて	17,980	1,712	剪斷
				45.2	横切りて	16,790	1,599	剪斷
				——	横切りて	21,500	2,040	剪斷
				37.9	横切りて	15,800	1,510	曲げ
	平均			42.8			1,715	
樺	總平均			44.1			1,749	
栗	42	遠江	95	34.3	横切りて	13,820	1,320	曲げ
				36.1	横切りて	12,850	1,230	剪斷
				——	沿ひて	10,790	1,030	剪斷
				36.1	横沿	11,270	1,070	剪斷
	平均			35.5			1,165	
栗	56	伊勢	66	39.3	横切りて	14,000	1,333	剪斷
				——	横切りて	14,500	1,381	剪斷
				39.3	横切りて	11,100	1,048	剪斷
	平均			39.3			1,354	
栗	38	秩父	98	33.0	横切りて	10,900	1,070	剪斷
				34.3	沿ひて	11,000	1,080	剪斷
				33.4	横沿	12,180	1,200	剪斷
	平均			33.6			1,117	
栗	總平均			35.7			1,206	
赤樫	5	常陸	35	42.1	横切りて	17,080	1,627	剪斷
				43.1	横切りて	17,930	1,708	曲げ
				41.9	沿ひて	18,000	1,714	剪斷
				43.7	横切りて	18,080	1,721	剪斷
	平均			42.7			1,692	
白樫	6	桶川	43	49.3	沿ひて	18,165	1,730	剪斷
				43.8	沿ひて	20,160	1,920	剪斷
				43.8	横切りて	22,000	2,100	剪斷
	平均			45.6			1,917	

第一表の三

材名	番號	産地	年齒	重量 一立方呎に付き封度	荷物の掛け方 年輪に、を	破壞總荷物 封度	剪斷の平均結局强さ 一平方吋に付き封度	破壞の狀況
檜	30	尾張	150	26.0	沿ひて	11,530	1,098	剪斷
檜	31	秩父	100	22.1	橫切りて	9,856	939	剪斷
				22.1	橫切りて	9,100	870	曲げ
				21.2	沿ひて	7,900	750	曲げ
				21.7	橫切りて	7,300	695	曲げ
				21.7	沿ひて	8,450	800	曲げ
	平均			21.8			811	
檜	32	尾張	109	22.6	沿ひて	8,890	845	曲げ
				23.9	橫切りて	9,350	890	剪斷曲げ
				23.9	橫切りて	6,850	650	剪斷
				26.6	橫切りて	9,980	950	剪斷
	平均			24.2			834	
檜	35	秩父	215	35.7	沿ひて	13,810	1,350	剪斷
				28.9	橫切りて	8,910	877	剪斷
	平均			32.3			1,113	
檜		秩父	29	——	沿ひて	9,970	950	剪斷
				——	沿ひて	8,400	800	剪斷
	平均			——			875	
檜		秩父	40	30.3	橫切りて	11,000	1,050	剪斷
				30.7	橫切りて	8,000	762	剪斷
				28.9	橫切りて	7,500	714	剪斷
	平均			30.0			842	
檜	總平均			25.8			882	
椹	24	尾張	95	23.9	沿ひて	6,670	635	剪斷
				21.8	沿ひて	6,100	571	剪斷
				21.6	橫切りて	7,330	697	剪斷
				21.1	沿ひて	6,420	611	剪斷
	平均			22.1			628	
椹	25	秩父	65	19.7	橫切りて	6,370	607	剪斷
				19.1	橫切りて	5,760	549	剪斷
				18.1	橫切りて	5,940	566	剪斷
				19.6	沿ひて	5,690	542	曲げ
	平均			19.1			566	
椹	26	秩父	60	22.0	橫切りて	7,700	733	剪斷
				21.4	橫切りて	7,020	669	剪斷
				21.6	橫切りて	6,930	660	剪斷
				22.7	橫切りて	7,240	690	剪斷
	平均			21.9			687	
椹	29	尾張	175	22.5	橫切りて	7,690	732	剪斷
椹	總平均			21.2			635	

第一表の二

材名	番號	産地	年齒	重量 一立方呎に付き封度	荷物の掛け方 年輪に，な	破壞總荷物 封度	剪斷の平均結局强さ 一平方吋に付き封度	破壞の狀況
縦	16	新宮	89	31.9	沿ひて	10,670	1,016	剪斷
				27.1	横切りて	9,930	946	曲げ
				27.2	横切りて	9,820	945	剪斷曲げ
				27.3	横切りて	10,210	972	曲げ
	平均			28.4			970	
縦	19	新宮	100	30.4	沿ひて	9,360	891	剪斷
				28.5	横沿	9,280	884	剪斷
				33.4	沿ひて	10,020	954	剪斷
	平均			30.8			910	
縦	33	遠江	100	27.1	横沿	10,320	1,020	剪斷曲げ
				28.4	沿ひて	9,600	938	斷剪
				28.4	沿ひて	8,240	820	剪斷
				23.5	横切りて	9,800	970	剪斷曲げ
	平均			26.9			937	
縦	34	遠江	120	28.9	沿ひて	8,480	835	剪斷
				28.4	横切りて	10,430	1,020	剪斷曲げ
				27.5	横切りて	10,000	979	曲げ
	平均			28.3			945	
縦	總平均			28.4			942	
杉	9	新宮	72	23.4	横切りて	8,000	762	剪斷
				23.7	横切りて	7,300	695	剪斷
				21.2	横切りて	7,600	724	曲げ
	平均			22.8			727	
杉	10	上野	131	24.6	横切りて	8,390	799	曲げ
				23.5	横切りて	8,930	850	曲げ
				25.2	横切りて	8,860	844	曲げ
				23.7	沿ひて	8,630	822	剪斷
	平均			24.2			829	
杉	27	新宮	71	23.4	横切りて	8,510	810	剪斷
				23.7	沿ひて	7,910	734	剪斷
				22.7	横切りて	7,950	757	剪斷
				22.7	横切りて	8,230	784	曲げ
	平均			23.1			771	
杉	28	上野	129	24,9	沿ひて	9,360	891	剪斷
				25.0	横切りて	9,370	892	剪斷
				24.8	横切りて	9,970	950	曲げ
				24.6	沿ひて	9,480	903	剪斷
	平均			24.8			909	
杉	總平均			23.8			815	

第一表の一

材名	番號	産地	年齒	重量 一立方呎に付き封度	荷物の掛け方 年輪に，な	破壞總荷物 封度	剪斷の平均結局强さ 一平方吋に付き封度	破壞の狀況
松	14	仙臺	70	32.8	沿ひて	11,360	1,082	曲げ
				35.8	横切りて	13,150	1,252	剪斷
				32.5	沿ひて	11,830	1,127	曲げ
				31.8	沿ひて	10,090	961	剪斷
	平均			33.2			1,105	
松	18	仙臺	79	31.2	横切りて	11,530	1,098	剪斷
				31.8	横切りて	11,860	1,130	剪斷
				33.9	横切りて	11,590	1,104	曲げ
				32.0	横切りて	10,940	1,042	剪斷
	平均			32.2			1,094	
松	39	遠江	131	34.8	沿ひて	12,450	1,180	剪斷
				45.6	横切りて	14,695	1,400	剪斷
				42.0	横切りて	15,970	1,520	剪斷
	平均			40.8			1.366	
松	40	遠江	186	32.5	沿ひて	11,340	1,120	曲げ
				33.9	横沿	12,770	1,260	曲げ
				33.0	横切りて	13,100	1,290	剪斷
				32.5	横沿	10,000	984	剪斷
	平均			33.0			1,164	
松	總平均			34.4			1,170	
栂椹	58	伊勢	110	29.8	沿ひて	8,700	866	剪斷
				29.8	沿ひて	9,230	913	剪斷
				29.8	沿ひて	8,430	837	剪斷
				30.7	沿ひて	8,380	831	剪斷
	平均			30.0			862	
栂	8	新宮	176	32.0	横切りて	11,980	1,141	剪斷
				33.7	沿ひて	11,620	1,107	剪斷曲げ
				32.2	横切りて	12,670	1,207	曲げ
				33.3	横切りて	12,990	1,237	剪斷
	平均			32.8			1,198	
栂	17	遠江	218	37.7	横切りて	12,500	1,190	曲げ
				35.8	沿ひて	11,740	1,118	曲げ
				36.1	横切りて	13,260	1,263	剪斷
				28.9	横切りて	13,400	1,290	剪斷
	平均			34.6			1,215	
栂	總平均			33.7			1,206	

$\frac{abc}{a'b'c'}$なる竪面積は各五平方吋なるを以て、剪斷に對する平均結局強さを一平方吋に付きf_s封度とすればf_sの値は

$$f_s=\frac{1}{10}\times(\text{破壊荷物、封度にて})$$

なるべし、然れども實際上剪斷面積は精密に五平方吋に製作せられざりしを以て、各試驗片に就き其寸法を度り、面積を見出して前述の平均結局強さを算定し第一表の第八行に載せたり。

試驗片には曲げ作用誘起せらるゝが故に中立面上に起る所の最大剪斷內力は、各剪斷面$\frac{abc}{a'b'c'}$上に分布せる平均內力の一、五倍に達し得べし、然れども此事實たるや等一切斷面を有する梁に對し且つ彈性極度以內の曲げ內力に對しては正確なりと雖も、實際の破壞試驗に在つてはf_sは略ぼ均一に剪斷面積上に分布せらるべし、隨つて第一表第八行に掲ぐるf_sの値は實用上安全なる數字と考ふるを得べし。

第一表によりて得たる平均、最小及び最大數量のみを摘出して第二表に掲げ實用に便にせり。

年輪に沿ひての剪斷強さと年輪を横切りての剪斷強さとの關係を知

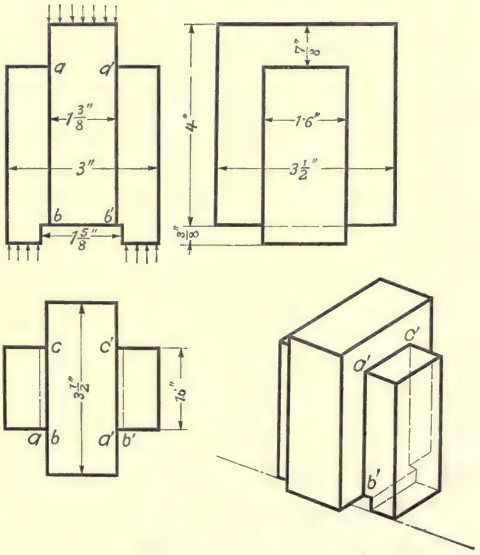
したり。（本報告書に記載せる試驗に先ち、東京帝國大學工科大學紀要第二卷第三號に述べたる豫備試驗を行ひ、其結果を檢察して多少の寸法變更を試驗片に加へたり、第一圖に今之を示す）

試驗を行ひたる試驗片總數百四十九個にして、內百十個は剪斷にて破壞し、三十二個は曲げ作用の如きによりて破壞し、七個は剪斷と曲げとの結合作用にて破壞したり。

各試驗片の中央部分を梁と見做し、試驗片の兩足內面間の距離を梁間（はりま）とすれば、梁の幅は梁間の二、一五倍に相當し、其深さは實に梁間の二、四六倍に相當す。斯くも寸法相應（ふさ）はぬ梁なれとも試驗總數の內一割七分は尙曲げ作用によりて破壞したり、斯の如き結果を得たるは是れ木材に在つては纖維に直角なる方向に於ける引き張り強さの格外に僅少なるが故なり。而して本試驗の範圍內に於て曲げ作用により破壞せし割合の最も多きものより順次に列記すれば鹽地、栂、樅、杉、檜、松等となる、故に此等の木材に於ては梁の深さを一層に大にする必要あるが如し。

剪斷作用によりて都合よく破壞し、曲げ作用による形跡少なきものは栂、樅、桂、楠、欅、檞、櫻等なり。

に竝行なる平面に於ける剪斷試驗を行はんとせば右の方法は何れも適當ならず、因て本報告者は次の如き極めて簡單にして容易なる試驗方法を曩に案出したり、之をあのくち剪斷試驗片と名づけ工科大學に於て實驗用に之を用ゐる。各試驗片は第一圖に示すが如き形狀及び寸法にして、木材の纖維は試驗片の竪方向 ab a′b′ と相竝べり。此試驗片を水平盤二個の間に置き上下より壓力を加ふるときは破壞剪斷面は荷物の方向に竝行なる竪平面 a b c a′b′c′ なるべきを豫期したり。偖此等試驗片を東京帝國大學工科大學實驗室に於ける Riehlè Bros. Testing Machine Co. 製造、四萬封度、複式槓杆型試驗機械の壓しつめ盤の間に置き、水壓ポンプを働かし、漸次に壓力を加へて試驗片を破壞



第　一　圖

十。

木材の剪斷試驗報告。

明治三十九年四月。
震災豫防調査會委員　工學博士　井口在屋合著
同　囑託　工學士　田中不二

〔震災豫防調査會報告、第五十五號、明治三十九年九月三十日。(1906)〕

剪斷試驗に於て必要なる條件二つあり、第一、試驗片に施す荷物の方向は剪斷する平面内にあるか、又は此平面に竝行なるべきこと、第二、曲げ内力を誘起する傾向なかるべきこと是なり。鐵や鋼の如き材料に在ては、試驗片を精密に作成してナックル接手の如き支持部に之を嵌め込むか、又は普通の打貫機械類を以て打貫試驗を施すか、或は又短距離に配置したる螺旋ボルトを以て試驗片を固着しボルトの間に剪斷力を加ふること等によりて前述の第二條件を滿足せしむるを得べし。

然れども木材に於て其年輪に沿ひ、或は之を横切りて且つ纖維の方向

業者一般には無論ふむきであること。規則條令法律條約などで定めた人爲的數量の變化は概して不連續であるが自然的變化、(例へば罐のなかの湯の自然にさめる割合、一年中の日日の數年平均溫度、日本人の各年齡に對する數萬人平均體量、ガスの溫度と壓力と比重との關係、など、)は一般に連續數量であり、なめらかな曲線でその在樣を明かに表はすことができること。直線で表はし得る變化、直線の方程式、碁盤紙を使用して直線式中の定數を定めること、有益な實例數件。

二次曲線、三次曲線、圓函數曲線、指數曲線、等を畫かき最大最小を求めること、變化の割合を求めること。任意方程式の實數根を求めること、例へば $x=100$, $x^3-3x^2+3=0$, $\theta-\sin\theta=\frac{3}{2}\pi$, 等の根。觀測數を用ゐて曲線を畫き方程式中の定數を求めること、例へば蒸汽機關、ガス機關、石油機關などのインヂカトル線圖から $pv^n=c$ と云ふ式中の定數 n と c とを求めること、等。複雜な品物或は工事の代價經費の實例三四件以上を知りて他の場合のを類推する法、等。

右の外有益な條項例題注意數件。

以上

が理會し易く且つ便利であること。二項式を概畧計算に用ゐること卽ち $(1+x)^n=1+nx$, 及び $(1+x)^m(1+y)^n=1+mx+ny$, 等。對數逆對數の說明、五桁對數表の使用法。透塡(すかし)め法(インタポレーション) $y_x=y_0+x\triangle y_0+\frac{1}{2}x(x-1)\triangle^2y+\frac{1}{3}x(x-1)(x-2)\triangle^3y_0+\cdots\cdots$ の應用。計算尺の道理及びその效用。雜問、例へば $y=mx+c$ に於て x と y との値二組を知りて定數 m と c とを求めること、$1\cdot8^x=4.2$ に於て x を求めること、$Q=\mu b h\sqrt{2gh}$ に於て h を求めること、等。

速さの說明、速さを測る法、單一弦運動の說明、實例、線圖、等。

圓、球、柱體、壕體、錐體の周圍、外面積、立積。錐臺、楔、類柱體、欵球、の立積。相似平面形、相似立體、及びその應用。類似楕圓を手はやく畫く法、甚だ大きな圓の一部分を畫く法、目分量で概畧な面積を求める法、それの間違の極度をあらまし見定める法。梯形規則、スィムプソン氏規則、及び他の概畧積分法。平均數の意味及びその求めかた。

圖法數學の效力の大なること、その應用の容易なること。いかなる問題でも幾んどみな圖法で解けるとて東京市の地圖見る樣な書物ができてをる、大は必らずしも小を兼ねぬ、精は必らずしも粗に優らぬ右の書物の樣に明細緻密な圖法はひまある學者には兎も角も我我工

けてをると我は信んずる、因て之を補ふ。幾何學三角法に於ては算術と代數とを無形道具となし、鉛筆定規物指コムパス分度器等を有形道具となして、二科目の或る部分は同時に、尋常進歩の速さよりも數倍速く、線角三角形圓圓函數等の重な性質を學ばせる。

求積法に於ては三角形圓楕圓球壔體錐體等に關する算術的代數的對數的幾何的三角法的及び圖法的問題を混用して教へる、又梯形規則スィムプソン氏規則等に據る所の概略計算法を教へる。以上が適切な方法の概略であるがこれを實施するに付ては實物標本圖畫線圖碁盤紙計算尺數量表製圖道具などが必要である。

とのべた、これが講義の前置きである。さて本講義の要目は次の通りである。

既知數のうちに粗畧な數があるときは精密な結果は得られぬこと。數の精粗は小數點の位置によらぬこと。工業上に必要な計算の結果は一般に二桁或は三桁あれば澤山である、隨てその運算中には三桁或は四桁を取扱へばよいこと。四桁に切り上げた數、それに乘除を施した數、及びそれの乘冪方根、の誤差。比例問題を割合の道理で解く方

ておくがよい、最大公約數最小公倍數などは一般な幼年初學者に容易にわからぬが天下の通義である。さてここで數量一般に關する公理を直ちに教へ、同時に等しいと云ふ符號＝の應用を教へるべし。割合や正比例や反比例や複比例や百分算や按分や何やかや盡く公理と＝とで一貫してしまうがよい。平方根立方根を求めることは算術でもできるができ合の表をこよみや汽車時刻表の樣に使はせるがよい。桁數五つよりも多からぬ對數表の使用法をはやくから習はせるがよい。代數に於てはむつかしい因子分解などは成るだけ避けて速かに方程式に移るべし。方程式中には小數を含んだもの、不盡數を含んだもの、などをも練習させる。指數對數二項式も必要である。三次式超越式の變化の有樣、最大な値、最小な値及び方程式の根を碁盤紙の上に曲線で表はさせる。

長さ廣さ大きさは普通教育の幾何學にくどい程綿密に說いてあり代數にその應用が澤山ある。然るに時間時刻時限速さ速まりは普通教育の數學書中どこにも明かに說かないでその例題のみをちらほら書いてある。これは對した不都合ではあるまいが全體を見通せば釣り合がか

筈。ニュートンは星學の研究に幾何的圓錐曲線を用ゐてそれから力學が發達した、それゆゑに工業の生徒は星學から力學に入らねばならぬと云ふ道理があらうか。……」

右 Perry 教授の改革案は英ギリス國の Science and Art Department に於て既に承認してをる。又同國の Battersea Polytechnic に於ても同じ方針で數學を教へてをる。この學校の講師 M. T. Ormsby と云ふは實用數學、土木工學、及び測量を擔任してをる人であるが一册の實用數學書を著はした、その大きさは縱八吋半、幅五吋半ページ數四百十、記載事項は算術、代數、幾何、三角法、立體幾何、求積法微分積分、である。斯の如く區別を立ててはありながらその說明證據運算等は全く舊來の法式を脫してをる、そうして全體を僅かに二か年で學ばせる。

工業教育中の必要科目として數學を教へるには右の樣な方法に從ふが最も適當であると我は信んずる。今我の意見が舊來法式と異なる所を次に述べて見えう。

算術に於ては並みの加減乘除から直ちに小數の加減乘除に移るべし、分數は最も簡單な二分の一三分のいくつ四分のいくつくらゐに最初は止め

實業教育の大改革を主張しておられる英ギリス國の John Perry 教授は次の様に言うてをる、「工業家のうちで工業問題に數學應用を一番上手にやる人が幾んどいつでも學校の教師教授など云ふ地位に落ち降り其事業の實際の有様には日日に遠ざかるかと思はれる、又教育の順序方法を編み立てる人は幼少な折りから我知らず無形的推理に得意であること家鴨はひよこから水を得意がる様に對した苦も屈託もなくやりあげた人人である、か様なあたまの人から見れば一か年かかりてもユークリッドの初歩がわからぬ小供は勿論愚物と言はねばならぬ。然るに本當を言へば無形的推理にたよりて事物を學び智能を練磨しえるは格外な腦力に限ることであり、しかもその腦力は恐らくは不健全なものであらう。小供に幾何學を教へるときユークリッド的理論をなぜ盡く放棄せぬ、ユークリッドの第一編第四十七をもこれは確實な定理であると言うて證明なしになぜ直ちに教へぬか、ユークリッドの第二編第五編の代りに簡單な代數二三ページをなぜ用ゐぬか、同じく第六編のうち澤山は公理であるとなぜ言はぬか。若し順序的推理が是非必要と言ふならばよくよく考へてみるがよい、本當に公理と言はれるものは一つもない

九。

實用 數學 摘要。

〔文部省 實業學務局 へ 報告、明治 三十五年 十月。(1902)〕

明治 三十五年 我(われ)が 文部省 の 命 を 奉じて 夏期 講習會 で 講義した 實用 數學 は 工業 教育 に 從事してをる 人 に 聽かせたのであり、その 大要 は 次の 通り で ある。

數學 を 教へる 順序 方法、今日 一般に 行はれてをる 所のもの、は 多くの 日數 月數 年數 を 不經濟 に 費やす、數學 は 頭痛 を 起す 學問 であると 幼年 生徒 に 思はせる、數學 を 學ぶ は 水練 の 理論、乘馬 の 理論、自轉車 に 乘る 理論、などを 學ぶと 同樣で 實用 には 無益な ものであると 或る 人に 思はせる。物理學 の 應用者 である 所の 工業家 に 取ては 最も 重寶な 最も 鋭利な しかも 無代價な 道具、數學 が 右の 樣に 幼年 子弟 の 貴重な 年月 を 浪費して 且つ 又 人に きらわれる とは そもそも なさけない 次第 である。

べし、若し又從來の如く二個の罐を使用しながら強き吸出し通氣法を用ゐたらんには充分なる燃燒を得て煙出しの頂部より黑烟を吐出すること無きも燃料の消費は自然通氣法のときよりも大なることあるべし、是れ通氣の速度過大となりて過分の空氣を供給するが故に瓦斯の溫度冷却する故なるべし斯る場合は無きにしもあらぬ譯です、石炭消費の點に於て最も上手なる火夫は幾分か黑煙を出すも亦同じ道理です

御話いたさうと存した所は大體此位でありまず、詰り私の考へでは大きな煙出しを造ると云ふことは少し馬鹿げてをる、因て前陳の如く人工的通氣法を用ゐんことを希望します

とですが押込みの方に於ては蒸汽罐の圍爐裡から罐の終尾に到るまで總ての部分に熱したる瓦斯が衝突する、普通船舶用の蒸汽罐の如き多數の細管を使用せる場合には細管と後部管板との接合際に熱瓦斯衝撃の爲め細管の周邊より漏を生ずる困難がある、吸出しの方に於ては瓦斯が繩若しくは鎖の如く引き張る力を受けるが故に瓦斯分子の運動は自然流下の水の如くである、故に前述の細管と管板の接合際などには幾分か靜止の有樣に近き分子が存在して居るべし

さて又右の押込み通氣法なり吸出し通氣法なりを從來自然通氣法を用ゐて居る所の蒸汽罐に直ちに應用して經濟上果して得ありや失ありや或は得失なきかと云ふに、蒸汽罐各部の割合等を考へず之れを用ゐるときは却て燃料の損失を生ずる場合が有ります、例へば今普通の自然通氣法を用ゐて二個の罐に適當なる空氣の分量を供給し得る場合ありとせん、然るに之を強き吸出し通氣法に變更したりとせば火皿一平方呎に付き燃燒する所の量は以前の倍若しくは倍以上に達するも差支なかるべし因て蒸汽罐一個のみを以て用を便ずることを得べし、然るに此くするときは石炭一ポンドに對しての傳熱面は減ずべし、隨て蒸發力も減ず

せぬ、併し全く考案なきにしもあらず其一二を申せば、煙道中に霧吹き機械を置きまして灰及び煤に濕氣を含ましめて其飛散を防くこと、煙道を横斷して網様の物數個を設け其下部を水溜に絶えず浸し置きながら其網を廻轉せしめる、煙道中に直立なる部分を設け其截斷面積を特に大にして之に煉瓦碎片、石炭滓を充たしめて之に水を注ぎ置く、先づ此位なる考に過ぎませぬ、茲に參考となるべき一件があります、近年大に發達進歩して來た所の瓦斯機關は其馬力數が百以上に達するときは機關使用者は自己用の瓦斯製造裝置を設くるを經濟とする譯ですが此瓦斯はダウソン瓦斯と稱へ無煙石炭若しくはコークを用ゐて簡單なる手順を以て製造し得る者で有ります之を機關のシリンドルに供給する前に充分洗濯する必要がある而して其洗濯法は前述第三項の方法に類して居ります蒸汽罐が咄き出す所の瓦斯よりして灰及ひ煤を取り去る必要のあるは專ら都市に於てである、都市に於てのみならば之が爲めに特別の裝置を爲すは亦已むを得ざる次第でありませう

そこで茲に注意すべき事がある、其れは押込み通氣法が宜しいか又吸出し通氣法が宜しいかと云ふ問題、何心なくふと考ふれば何れも同じこ

き九十呎の速度を使用します、左れば今程申したる大形ランカシヤ蒸汽罐三個から排出する所の瓦斯一秒に付き三百立方呎を通過せしむるに必要なる管の直徑は二十四吋四分の三となります、一秒に付き百立方呎を排出するには直徑十四吋四分の一の管を要す故に人工通氣法を用ゐますれば流量頗る大なるも瓦斯を導くべき管或は樋はさ程大ならざるべし

扇風機は格別大きくする必要はありませぬ、別して近頃では瓦斯機關石油機關などを使ひますれば高き廻轉速度を得ることは難くありませぬ、隨て所要の扇風機は小さな者て事足る筈です又燒け損じる困難に對しましてはエリス、エンド、イーヴスの方式を用ゐるか、さもなくば之に類する方法例へば普通使ひ居る所のエコノマイザなどは瓦斯の溫度を低くするのみならず亦大に燃料節約の効あるは勿論です、今一つ殘つて居る所の事柄にはチツと困りますが何とか方法が有りさうなものと思はれます、燒け殘りの灰及煤の始末でありますに付ては簡約にして有効なる方法は未た嘗て聞知せざる所です且つ又專ら實驗的研究を要する事項であるに由て只今は何たる纏まりたる考案をも所持しては居りま

氣法を用ゐるときは化學上必要なる分量の一、二五乃至一、六五倍にて充分なるべし人工通氣法、自然通氣法何れにしても煙出しの馬力は機關の馬力百分の一或は百分の一以下のものでありまず、是ればかりの仕事をするが爲めに廣大なる煙出しを造るは如何なる理由であるか疑はざるを得ざる事柄です、就て何か機械的困難は無いかと尋ねまするに先づ第一番に困難と思はるる點は一秒時に付き三百立方呎と云へる可なり大なる流量を輸送すべき管或は煙道を敷設するを要す、さて煙道内に於ける瓦斯はどの位の速度を以て絶えず進行しつつ有るかを知るべき算式を求めて計算いたした結果が茲にありますが、此算式は略しまして其結果だけを申上ませう、但し是は決して想像的のものではありませぬ、百尺の煙出しに於て溫度、截斷面等の適當なる値を施して計算すれば速度は一秒に付き凡そ二十五呎と成る、高さ百五十呎の煙出しに於ては一秒に付き凡そ三十呎高さ二百呎のに於ては一秒に付き三十五呎、高さ二百五十呎のに於ては一秒に付き四十呎位です、然るに今扇風機などを使ひ人工的に瓦斯を押し遣り或は引き出す場合に於ては瓦斯進行の速度は一秒に付き八十呎乃至百呎に達するも困難はありませぬ、今假に一秒に付

者は徑が八呎、長さ三十呎、長さは長くとも短くとも對した關係はありませぬ、煙管の徑は三呎三吋、さうして火皿の長さ六呎、因て其面積三十九平方呎卽ち概略四十平方呎あります、火皿の面積一平方呎に付き一時間に付き石炭十五ポンドを燃燒せしめ又自然通氣法の場合には石炭一ポンドに付き空氣二十四ポンドを供給し人工通氣法の場合には十八ポンドを供給すとして計算を致せば右のランカシヤ形蒸汽罐では自然通氣法を用ゐるとき一秒に百立方呎の瓦斯を吐出さなければならぬ、今斯の大きさの蒸汽罐三個を絶えず使用して居る工場に於て其煙出しの通氣力を充分に強く見積り水嵩二吋と假定して前の算式に據り煙出しの馬力を計算しますれば其結果は僅に六馬力と成ります、さて此三個の罐に適する蒸汽機關には一時間一馬力に付き石炭の量三ポンドと見積りて相當と存じます、されば機關の馬力總計は六百と成る、之に由て觀ますれば煙出の爲しつつ有る所の仕事は機關の仕事の僅に百分の一に相當いたします

ところで此計算には化學的化合の爲めに必要なる空氣の分量二倍を供給すると假定し右の結果を得ました、併し押し込み若しくは吸出し通

Hも亦相等しとして勘定して試ます一つの形が出て來ます所で煙出し外の空氣の溫度を華氏寒暖計で凡そ六十度と假定しますれば垂直に下向なる部分中の瓦斯の溫度は八十度位なることを要します、若し此溫度を低くして外氣の溫度六十度まで冷却することが出來たならば同一の高さを有する上向の煙出しよりも通氣力が強いと云ふことは明です、それから普通の煙出しが爲して居る所の仕事の割合卽ち馬力は誠に僅なる高であつてえらい金を掛けて造る程の用もないかの樣に思はれます、煙出しの馬力は次の如し

$$HP=\frac{62{\cdot}4\,h\,V_0\,W}{12\times60\times33000}\times\frac{t_1}{t_0}.$$

小さいhは煙出し內の瓦斯の通氣力を水の嵩吋にて表はしたる者、Wポンドは一時間に燃す所の石炭の分量、其他は都て前式の通りであります、そこで通常陸に使つて居ります蒸滊罐の大小をざつと申しますと、一番小さいので最も普通なる者はコルニシユ形、徑三呎九吋、長さ十呎煙管の徑二呎、火皿の面積八平方呎、此位が最小にして且つ普通なる者です、稍く大なる蒸汽罐を要するときはランカシヤ形を用ゐる、此形の最も大なる

煙出しに於きましては

$$h=\frac{12}{62\cdot4}\left\{\frac{t_0}{t_2}\left(\cdot0807\right)-\frac{t_0}{t_1}\left(\cdot0807+\frac{1}{V_0}\right)\right\}H.$$

此小さいhは煙出しの通氣力、通常水の嵩さ幾吋を以て勘定する分量、t_0は氷點溫度、t_1は煙出し内の瓦斯の溫度、t_2は煙出し外の空氣の容積を溫度氷點に於て立方呎にて量りたるもの、それから大きなHは煙出しの高さを呎で表はしたるもの、溫度は總て華氏目盛に據り測りたる絶對溫度です、煙出しの高さHが二百呎なるも其通氣力hは水嵩にて僅に一吋半ばかりです、さて又下向に瓦斯が流出する煙出しに於きましても全體の式が殆ど右の通りであります卽ち

$$h=\frac{12}{62\cdot4}\left\{\frac{t_0}{t_3}\left(\cdot0807+\frac{1}{V_0}\right)-\frac{t_0}{t_1}\left(\cdot0807+\frac{1}{V_0}\right)\right\}H.$$

t_3とは垂直に下向になつて居る部分内の瓦斯の絶對溫度、さて上向の煙出しと下向の煙出しとの二つに於て通氣力が同しくなる場合には右二つの算式に於てhの値が互に等しくなれば良い筈です、さう致して高さ

時幾百萬、先つ斯の如き有樣でありませう、そこで第一の關門の幅をば六七間ばかりに擴張改正したる譯であります、斯改正を施したる細管をセルヴ管（Serve Tubes）と謂ひ其內面を吸熱面と稱し外面を放熱面と號します此方便と今程申しましたる通り圍爐裡に供給すべき冷めたき空氣を豫め熱する方便とに因て煙出し內の溫度が餘程低くなり扇風機を用ゐて故障なく、蒸汽罐の效率も亦大に增加した譯です

是で三通に大別したる通氣法の各部類をも先づ申上げた積です、さて通常の陸地用蒸汽罐に對しての通氣法は我々總て知つて居る通り、丈けの高い煙出しを設ける、それで五百馬力以上千馬力位の蒸汽罐に對しては頗る大なる煙出しが必要である、地盤の惡しき處に於ては煙出しのみに費す所の錢の高が實に對した者である、近頃は鍊鐵の煙出しも幾分か出來始めた樣です鍊鐵製であるときは直段は餘程に廉に出來ませう、併しながら抑も普通の煙出し通氣法と云ふ者は如何にも機械的の方法でなくして誠に所謂單純なる考案であつて效力が其大さ及び費用に比較しては薄いものである、丈の高い目形の重い煙出しを造らずとも外に良い通氣法があらうと思ひます、それは如何なる譯かと申しますと、普通の

に作り附けに拵へてある、之を以て傳熱面を增加し隨て瓦斯の溫度が幾分か低くなります、茲に傳熱面と云ふ申分は少々分り悪い、卽ち此細管の外側には水があつて內側には火卽ち熱したる瓦斯が這入つて居る、それで其水に觸れて居る所の面積を取るべきか、瓦斯に觸れて居る所の面積を取るべきかと云ふ事柄が分り悪い、從來此點は判然して居りませぬ、と云ふ理はどちらを取りても其面積の差は僅になる故である併し今茲に傳熱面が熱を傳達する順序を言ひますれば頗る差がある、瓦斯と金屬と相觸れて居るときには其傳熱力は大變小さい、水と金と觸れて居るときには傳熱力が大きい、瓦斯の場合は水のに比較して四割乃至六割位であらう、事に依てはまだ小さいかも分りませぬ、因て此エリス、エンド、イーヴス式では傳熱力の悪い部分卽ち瓦斯と金屬と相接して居る所の面積を大きくした理です、今之を平易に申して試ませう、總て瓦斯體から水へ熱を傳達するには通常其間に金屬板を使ふ金屬板を用ゐるは熱と云ふ通行人に對して二個の關門を設けて通り拔けの自由を制限すると一般なる事柄です、瓦斯町から金屬町に入る處に於て關門一個此幅三間、又金屬町から水町に入る處に於て關門一個此幅六間、然して通行所望者は每秒

斯で煖める、さうしますと煙出し中の瓦斯の溫度は餘程低くなる此式に用ゐる煙出しは極く短くして建物の屋根から僅に數尺突出して居るのみです、此吸出し通氣法は餘程以前から使つたら宜からうと云ふ考は勿論あつたので幾度びもやつたことはあるけれども併し成功しなかつた場合が多い、それは熱したる瓦斯排出のため普通の往復ホムプを使用しては無論嵩張りて始末が附かぬ、兎に角廻轉ピストン送風機又は渦巻き扇風機を使はなければならぬが渦巻き扇風機を用ゐますれば熱したる瓦斯の爲め機械の各部が漸々燒け損じると云ふ困難がある、卽ち羽の尖り端が第一番に損じ軸心棒の承は溫度高きが爲め油の循環自由ならずして燒ける、斯う云ふ困難があるこれは普通の煙出し內の瓦斯の溫度が凡そ華氏六百度位になる故である然るに右申しましたるエリス、エンド、イーヴスと云ふ方式では圍爐裡に供給すべき空氣を暖める、因て華氏溫度六百度位であつたものが凡そ四百五十度位に低くなる、其位に至りますれは餘程フアンの困難は少くなる、それと同時に此式に於きましては（少し問題外になりますが）蒸汽罐の小管卽ち煙を通過せしむる細き管の內面に於て管の長さに平行したる骨板とも申すべき者を八枚つゝ各の管

火皿の下なる灰の落ちる部分のみをば全く密閉して仕舞ひます、之に管を取り附けて壓力のある空氣を送る、是も船舶用の蒸汽罐に偶まには用ゐる今一つは壓縮したる空氣では無く蒸汽にて空氣を誘導して其れを火皿の下から送込む、斯う云ふ方法がある、是は今日本にも二三個所あると思ひます、九州の三池炭坑、それから大阪のたしか電燈會社にも使つて居ります

第三類は吸出し通氣法卽ち英語でサクシヨン、ドラフトと申します、是も矢張細分することが出來る、其一は機關車の場合のやうにすたり蒸汽をば使ひまして、それを細く短い煙り出しの中に導き強い速度を以て石炭燃燒に由り生ずる所の熱したる瓦斯を吹き出し之と同時に火皿の下から空氣を吸込む方法、又一つは煙出しの或部分に通風機械を置きまして人工的に煙を吸出す方法、是は餘り澤山は使つて居りませぬが、幾分かあります、此方法で極く能く成功して居るものにエリス、エンド、イーヴス結合通氣法と云ふのがあります、此式に於きましては今申しました通り煙出しの下部に渦卷き扇風機を備へ小形蒸汽機關で之を廻轉して熱したる瓦斯を吸出すと同時に圍爐裡に供給すべき冷たき空氣をば其熱瓦

繰り返しにしたやうになりまして、冷い風のある所はサイフォンの長い管に相當し、熱した風のある部分はサイフォンの短い管に相當する、斯う云ふ譯で風を引き出すことが出來るのであります、是は例のガス、プロヂユーサルと云ふ種類には實際に使つて居りますガスプロヂユーサルに於きましては瓦斯を導く竪なる部分は十尺乃至二十尺位、然して後水平なる部分を通る、是は成るべく長い方が宜しい數百尺もあつて宜しい位です、此仕掛を使ひまして竪向に下に下つた所の風の溫度と外氣の溫度と同じくすることが出來たと假定すれば並の上向に流れる煙り出しよりは引きがよくなる譯です、是は後にちよつと御話致しませう

今度は第二類の押し込み通氣法、是はざつと先づ三種に分けて宜しいかと思ひます、併しまだ此外に幾分か種類の變つたものがあるかも知れませぬ、私が考へます所では三種らしく思はれます、其一つは船舶用の蒸汽罐で度々使ひます仕掛です、火焚き口並に灰の落ちる所の全體を密閉したる部分にして仕舞ひます、卽ち火夫が密閉したる室内に在つて其室内の壓力を普通の大氣の壓力よりも幾分か高くして居る因て其高い壓力で火皿の下から空氣が這入つて往きます、之が一つの方法、次の一つは

扨て其圍爐裡に空氣を供給し且つ出來得る所の瓦斯を排出すと云ふ方法は蒸汽發生の爲めに用ゐる場合に於きましては之を大別しまするとと三通りになるかと思はれます、第一類は自然通氣法、第二類は押し込み通氣法、押込みと云ふ言葉は不適當であるかも知れませぬ、之を英語で申しますればフォルスド、ドラフト、第三類は吸出し通氣法、英語サクシヨン、ドラフト多分此三通りに止まるであらうと思ひます、さうしまして此三通の中を又一つ宛細分することが出來る第一類の自然通氣法では普通の陸用蒸汽罐などに使ひまする上に流れ出す煙り出し、今一つは下向きに流れ出す煙り出し、下向きに流れ出す煙り出しと云へば極く稀でありますが全く無いことも無い譯であります、此は如何なる仕掛かと申しますれば通常の上向の煙出しの如く直立したるものを造り其頂上から水平なる煙道を造り、其水平なる煙道の終りに於て又垂直に下向になつて居る煙道を造る、斯う云ふ仕掛をして此水平なる部分を冷す其冷す方法は大氣に曝して置いても宜しうございますし、又水で冷すことも出來ませう、兎に角水平なる部分を冷す、左樣致しまするとと云ふと垂直に下向なる部分にては其中の瓦斯の溫氣が冷くなる、因つて丁度サイフォンを引

八。

圍爐裡の通氣法。

〔工學會誌 第二百五卷、明治三十二年二月。(1899)〕

今夕御話致しませうと存ずる事柄は圍爐裡の通氣法と云ふのでござります、圍爐裡と云ふ文字は或は不適當であつたかも知れませぬが私の意味では石炭、薪などを燃すが爲めに四方上下六面の中凡そ四面丈けを取圍みまして一面からは空氣を供給し殘りの一面からは燃燒に因つて出來たる所の瓦斯を放出すと云ふ裝置のものを指したのでありまず、卽ち開いたる爐でなく圍んだる爐である、さう云ふ爐の中に於きましても特に蒸汽を拵へるが爲めの爐の事を御話する積りであります

蒸汽を拵へる爲めの爐の外には勿論、冶金術などに使用する爐は色々種類も澤山あることでござりませうが是は私の今夕申上げやうと云ふ事柄の外であります

喞子、曲柄、制動機、衝頭接合、梁の彎曲、軸頭包金と譯したりとも前者に比較して更に何等の便益なきのみならず却て解し易からざるの虞あり、然るに茲に又說を爲すもの有らん曰く通常の文書中に假名書きの術語を用ゐ然も「はらむて」「ぽんこつ」「まるとん」等の如き俗語を其間にはさむときは西洋服を著け下駄をはきたると一般にして大に文章の威嚴を損す強て漢字を用ゐるは之が爲めのみと、是れ未だ熟考せざる者の一說のみ、抑も言語には通俗にして野鄙ならざる者あり、通俗ならざるも野鄙なるものあり、野鄙は勿論避くべしと雖も之と共に通俗語を擯斥すべからず、口に唱へ難き漢音又は口に唱へ易きも耳に混じ易き單音は畢竟吾邦の術語としては其普及覺束なし、是亦優勝劣敗の理なり、故に余輩は前述四ケ條の方針に基き工學上の術語を選定せんことを希望す

其他の七十六字、百〇四字等それぞれ皆同一なり、船頭多ければ船も山に登らん、同音異義の文字多ければ山或は海とならん、水或は火とならん、之を要するに單音なる漢語は後來若し普通一般に行はるる術語となりたりとも其不便實に大なるべし例へば蒸汽罐を略して汽罐と書き之を音讀するときは機關と全く同聲にして實際上誤解を生ずること往往これあり

術語を撰定するには如何なる方針を取るべきか、曰く解し難き漢字を成るべく避くること、曰く他音と混じ易き單音を避くること、語路あしき音を避くること、平衡は平行と混じ、拱は強と混じ、人孔は人工と混じ、火管罐、彎曲力率は語路あしきが如し、曰く日本固有の言語及び普通の俗語を用ゐること、曰く歐羅巴及び亞米利加の諸國に普通行はるる所の術語にして日本語中に之をはさみ込み其語路よろしき者は強て譯語を附せざること、今玆に二三の例を擧げん、Arch は「せりもち」、Piston は「ピストン」、Crank は「クランク」、Brake は「はどめ」、Butt joint は「毛抜接手」、Deflection of beam は梁の撓み、Bush は「かりも」、と譯すれば音平調にして語路あしからず口に唱へ耳に聞きたるとき他の言語と間違ふることも無かるべし、然るを拱、

書に據らざれば發音すら爲す能はず、是れ適當なる術語を撰定するは實に困難の業なるが故なり、學術技藝の進歩發達を圖らんには其術其藝に習熟せる學士、技師、工手、職工等多數の人を養成すべきは更に論なし、之を養成するは斯道を普く世間に及すと同一なり、偖て學術技藝を山間、海濱、都會、田舍の差別なく自在に運搬する所の鐵道機關車とも云ふべき術語は成るべく平易にして口語に通じ易からんことを期すべし、難澁解すべからざるは鐵道貨物停滯するに似たり、術語の混合、發音の出來ぬ等は運輸課の事務不整頓にして荷物の混合、送り出しの間違あるに彷彿たり、元來單音なる漢字は支那本國にては平上去入の四聲（現今の官話は上平、下平、上聲、去聲）に區別し且つ日本言語中に存在せぬ音を頗る多く有することなれば單音なる譯語を用ゐるとも決して不都合ある筈なし、吾邦にては然らず、試みに之を言はん、こうなる音を有する普通漢字は凡そ三十三、かうは六十六、かふは八、くわうは十八、合計百二十五字、そうなる音を有する普通漢字は二十六、さうは四十七、さふは三、合計七十六字、しようなる音を有する普通漢字は十七、しやうは五十一、せうは三十二、せふは四、合計百〇四字而して口語にてはこう、かう、かふ、くわうの百二十五字皆同一なり

七。

術語撰定の要旨

（明治三十一年七月機械學會正員會に於て述ぶ。（1898）
機械工學術語集、第一輯。明治三十四年十月）

言語文章は思想を運搬する鐵道機關車なり、現時工學、工藝上の術語にして普く技術者の間に行はれ一たび之を聞くときは其意義釋然たる者十中蓋し一に及ばず、歐羅巴及亞米利加の所謂文明國に通用して居る所の術語に解し惡き漢字譯を附けて之を文書に記載し對面會話の際には原語を有の儘に使用するは文明國教育の涵養を受けたる人々の一般なる習慣ならんと思はる、筆にては洋燈と書き口にては「ランプ」と云ひ、筆にては護謨と書き、口にては「ゴム」と言ふが如し、其他喞筒、軌條等、適例尚ほ多かるべし、然れども此等の數例は今日に在りて既に普通一般の術語とも云ふべき者のみを擧げたるなり、否らざるものは或は難澁解すべからず、或は單音にして他の語と混じ易し或は普通用の漢字に非ざるが故に字

聞紙でたびたび見ることあり又人の言ふのを聞くこともたびたびあり是れ甚だ不都合なる語なり何となれば距離と時間とのみの關係を示す所の事件を速力と言ふは酩酊と言ふべきを酩酊酒と云ふが如くなればなり

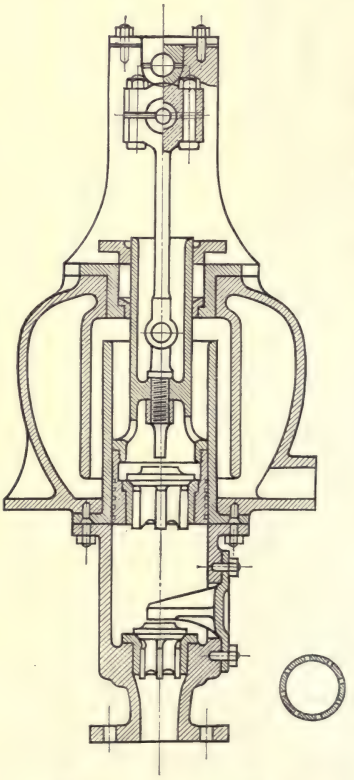
四。

はやさ、速度。

〔工學會誌 第二百二十二卷、明治二十五年二月。(1892)〕

Thomson and Tait 兩氏の Natural Philosophy に云く The rate of motion of a point, or its rate of change of position, is called its velocity. It is greater or less as the space passed over in a given time is greater or less: and it may be uniform, *i. e.*, the same at every instant; or it may be variable. 物理學術語對譯字書に云く Sokudo; Velocity; Vitesse; Geschwindigkeit, die. 英華字典に云く Velocity, *n.* Speed, 快、疾、快者、速率 the velocity of the wind, 風行の快、是等は皆定義に適ひたる譯語なるべし　大槻文彦先生の日本辭書言海に云く　サ(接尾)(狀の略)　形容詞の語根に添ひて其語を名詞とし其狀態程度をいふ語「赤さ」「廣さ」「重さ」「遠さ」「嬉しさ」「悲しさ」「淋しさ」因て余は今 velocity を邦語に譯して**はやさ**と言ひ漢字にては速度と言ひて適當ならんと思ふなり然るに近頃速力なる語を雜誌類や新

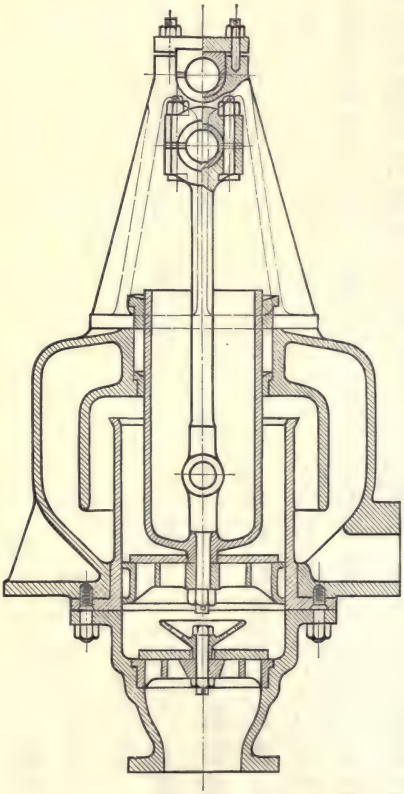
にピストンとプランヂャルのやうであるが側路がない此周圍の筋は單に數條の筋を付けるのみで水の漏れを防ぐ爲めパッキングなどを入れないやうにする其れから此處（第十一圖畧す）にありますのを一寸申しますがバッケットの取り外しを便利にしたるものでありまして昨年博覽會に出しました後に改良を施しましたれば更に便利を增しました其圖は此處（第十一圖畧す）にあります先づ申上げますことは此位でありますが尙ほ諸先生方の御考を伺ひたう御座ります



第十圖

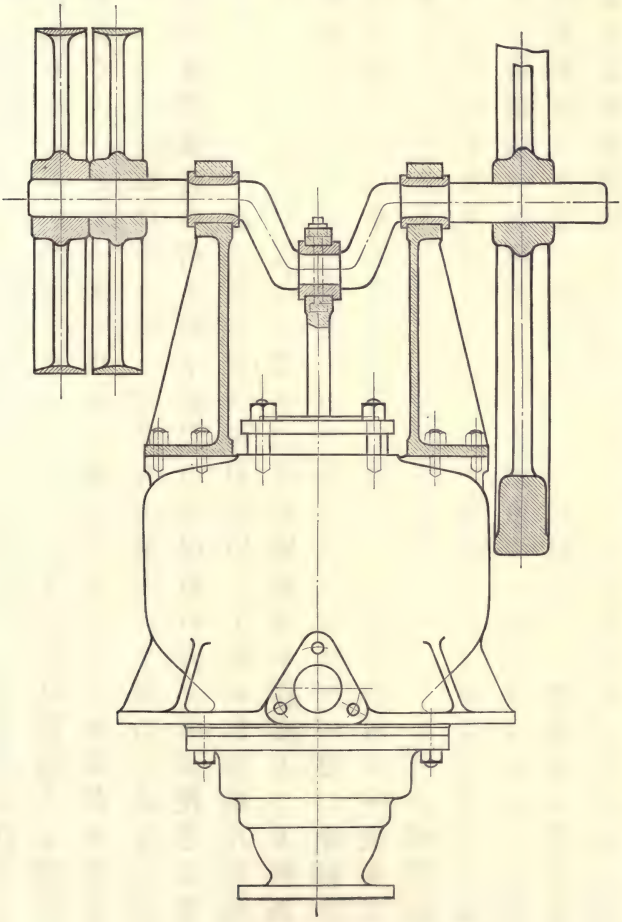
けてある是(第五、六圖畧す)れなどは此處に持つて行つて空氣室を別に拵へてある其れで私のではプランヂャルの周圍を空氣室となして其直ぐ上にクランク軸の受けを備へてあります依て機械力で運轉致しても宜しう御座ります或は小さな喞筒で水量が少ないならば人足の手で運轉致しても差支ないけれども元來其護謨製又は革製の瓣は斯う云ふ風に使ますとどうも傷み易くて瓣の取換が面倒なる場合がある例へば蒸滊機關の空氣喞筒の瓣の如きも護謨で拵へる其受る所の壓力は極大きくなるとも十五磅即ち水の高さに直して三十四尺位其でも可なり傷むから時々取直す、依て此の小生の意匠に係る喞筒は水の高低大なる所には幾分か不便である、就きましては是れと同じやうな形に致しましてプランヂャルの下に於て高壓力の爲に傷まないやうな瓣をつけたいと種々工夫いたして遂に其れを考へ出しましたのは此處にあります(喞筒筒構造の模型を示す第十圖特許出願中に係る)是れが喞筒の筒です此處に瓣が付いて居る是れはピストンではないがピストンのやうなもので瓣が傷むと云ふ恐れが先づ少ない併し砂が這入れば多少傷むが其場合は仕方がない兎に角壓力の高い所、高低の大なる所に使つて宜しい構造は實

處（第九圖）にありますのが此種類の喞筒で私の意匠に成りたるもので昨年の第三回内國博覽會に出品して二等進歩賞を得たる者です先づ下の方から水の道に付いて申すと吸込管は下から眞直に付て居る此處に吸込瓣が付て居る護謨又は革で作る其直ぐ上に出口の瓣が付て居る是も亦護謨又は革で造る斯うなつて居から水の迂廻は極めて少ない下から垂直線なる道を通過して水が出口の瓣の上に出て仕舞ふ其後は一向構はないさて又此處にプランヂャルが付て居るから往復とも同量の水を出すことが出來ます又空氣室と云ふものが一般に入用でありまして大抵は別なる處に設



第九圖ノ二

所にするか或はプランヂャルとピストンと一所にすれば宜しいピストンとプランヂャルとを一所にするときは特に側路を設けて水を迂廻せねばならぬ且つ又此ことは餘り容易くありませぬバッケットとプランヂャルとを一所に致すときは極都合能く構造が出來ますそこで此



第九圖ノ一

殘つて居るゆゑに此場所に空氣が這入ることが出來る所でこれは小さな場所ではありますが御覽になると分りますが空氣が這入る場所はなか〳〵大きくあります一ストローク毎にプランヂャルが利かざる容積は殆と其十分の一位はあるやうです先づ假りに十分の一あると考へて見ると此場合で吸込の高さが十五尺あり出口の方が六十尺であると假定致しますと云ふと此プランヂャルが退いたときは其空氣は大氣の壓力のざつと半分位までになる其位空氣が膨脹する膨脹したる後吸込の管の瓣が開く其れから其大氣壓の半分になつたやつが更に壓縮されなければ水が出ない只今六十尺と假定致しましたから四分一位まで縮まる元來プランヂャルの十分の一の容積があつてプランヂャルが引込んだときは十分の四となり、プランヂャルが飛出したときは十分の一になつて仕舞ふから畢竟一遍出たり引込んだりする毎にプランヂャルの十分の三だけ水を汲むことが出來ないから大變まづい喞筒になります、因て此空氣の入る場所は成るべく無いやうに致さなければならぬし且つ又小さな喞筒でありましては成るべく構造の容易いやうに致して往復共水の出るやうにしなければならぬからプランヂャルとパッケットと一

と相等しと假定致します、さう致しますと云ふと2·66×15h（hは卽ち吸込の管の直立の高さ）是れだけになります此外に今の水蒸瀲の壓力と瓣二つを揚げるに要する壓力と相合し一平方吋一磅半と假りに致しますさう致して計算致して見ますと吸込の管の長さは十八、二呎是れだけしか利かない此外管の据付けがまづくて空氣の這入る所があると云ふ場合には更らに短くなります其れから吸込管中の水の平均速度二百五十呎でなくてもうちつと大きくなると吸込の高低は是れより尙ほ少なくなる又管の中に垢が溜ると摩擦が澤山になるから十五呎位で利かないかも知れぬ十五呎でも吸込が利かないと云つて喞筒が利かないと稱るのは聞えない一々理由を計算して見ると彼の位に成る筈であります

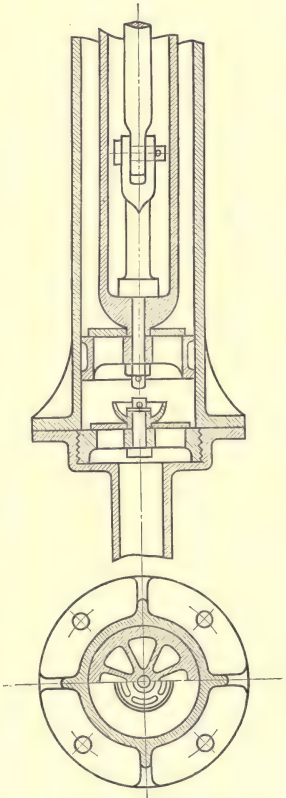
茲に今一言申しますことは喞筒の構造に依つて水の出が澤山なると澤山ならぬとの善惡がありますが其中一番間違ひ易ひ點は吸込の瓣と出口の瓣との間に空氣の這入ることの出來る場所を拵へてあることです態々空氣場を拵へは致しませぬが遂いどうかしてさう云ふ場所が出來る、さうすると喞筒は誠に工合が惡い例へば此處（第四圖）にあります喞筒を見ますと今此位置ではピストンが右に來て居る然る所僅の場所が

は水蒸瀘の壓力が幾分かありますから三十四尺に達することが出來ぬ夫れが一つ其れからして吸込の瓣を開くには幾分かの力を要しますから夫れが三十四尺の中幾分か要する其れから吸込の管の終りに塵除けが付いて居る其穴を通るとき幾分か抗抵をなす又其塵除けの傍に於て水の後戻りを防ぐ爲に瓣が付いて居る其れを開くに幾分か高低を要する又管に曲りがあれば高低を要する其から急に管の大きくなる所あれば高低を要し其れからして管の中を水が通るに摩擦に打勝たなければならぬ其れが爲に幾分か高低を要する其れで今此處に一例を取つて申上けて見ませう吸込管中の水の平均速度を一分時間に二百五十呎と致しますれば先刻申しました計算を假に誠として最大速度は一分時間七百八十五呎卽ち一秒時間十三、一呎になります其十三、一呎の速度を生ずるが爲めに二、六六呎の高低を要す其れから吸込管の入り口に於ける高低の損失、曲り一個に由て起る所の高低の損失、急に大きくなる所に於ての高低の損失、此三つだけを總計致しますと云ふ $2.66 \times .905$ 卽ち二、四呎是れだけの高低が入用其れからして管の中の摩擦こいつは管の寸法を極めなければ分りませぬから假りに管を二吋徑と致しまして長さと高低

のと殆と同じですプランヂャルが降るときにバッケットの瓣が開き吸込の瓣が閉ぢます併ながらプランヂャルが退けたゞけの水が上に昇らなければならぬ今度はバッケットが昇ると吸込の瓣が開き筒とプランヂャルの間にある水は出なければならぬ是れで往復とも水を出す事が出來るそこでプランヂャルの截斷面積を筒の截斷面積の半分に計算さへ致して置けば往復共に同仕事をする譯になります併しながら吸込の管が長くして出口の方の管が甚だ短い時は或は平等にならないかも知れぬ、是れは格段の場合であります

是れ等の種類に限らず總て喞筒として幾分か困難を感ずる所の點を擧げて見ると云ふと先づ第一番に水面から幾尺の所に於て能く水が上がると云ふことで能く言ふことであるが普通先づ十五尺位或は極大きくて二十四五尺位でありますと云ふのは本にも書いてあり人も申します何故其位になりますか愈上からないと云ふことの理由があるかを調べて見ませう先づ第一番に吸込の瓣から下は水が若し動かないものであれば三十四尺位は出來ると云ふ譯でありますが兎に角水は動く水が動きますと其れが爲に幾分か抵抗を要します其れが一つ其れから今度

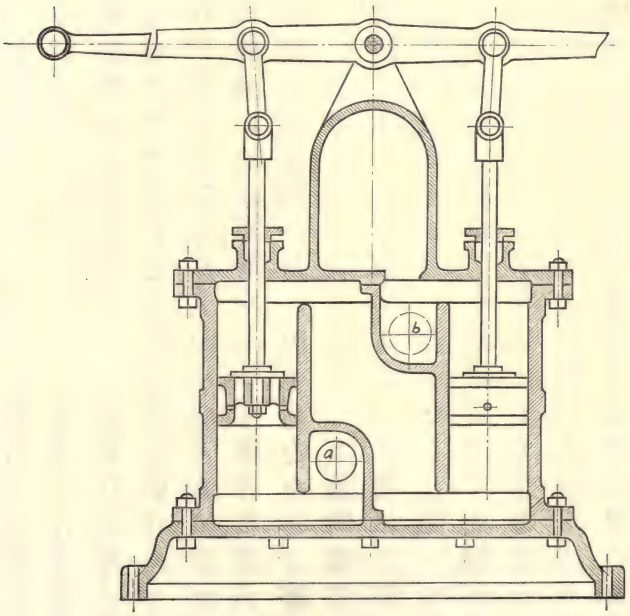
又瓣が二つ、筒が一つのみあつて往復共水を出す事が出來る喞筒の種類はざつと申すと二種あります第一プランヂャルとピストンと一所にしたものが此處(第四圖)にあります(少しくどくなるかも知れませぬが申上げます)吸込の瓣を開けて水がピストンに付いて行くとプランヂャルの周圍にある所の水は行き所がないに依つて脇道に行つて出口の瓣の上を通つて此方に出て行きます今度は此ピストンが左りから右に行くと考へます然る時は吸込の瓣は閉ぢて仕舞ひ出口の瓣が開きピストンが押し出したる水の一部分は筒とプランヂャルの間に這入つて行き他の一部分は汲出される斯う云う譯で是は筒が一つ瓣が二つで往復とも仕事が出來る第二種はバッケットとプランヂャルとを一所に組合せたもの卽此(第八圖)が一例です其働きは只今申した



第八圖

が二つありて瓣も二つあるのみで例の複働喞筒と同じことになります（注意、第七圖に示せる者は二瓣の間に空氣の潛伏すべき場所あり太た惡しき構造と云ふべし用うべからず）

是と同じやうな構造でありまして二つの瓣が同時に上つたり下つたりするものがあります然るときは一つは上向きに開きますさうすると今申したのと同じ働になり往復共水を上げることが出來まして矢張瓣が二つで筒が二つである



第七圖

ある其れ位に拵へても平均速度の十分の一までは確かに不同がある、ぢやに依つて成るべく此今申した速度の不平均と水の速度の不平均を防ぐ爲めに成るべく平等に仕事を爲し得る喞筒が欲しい平等に仕事をせんとなれば先刻申上げました複働喞筒を使へば宜しい之なれば最大速度と平均速度との割合はずつと減つて半分になるから宜しいけれどもさう致すと瓣が四つ付かなければならぬ隨分込入つて參ります其れからしてプランヂャルばかりでは勿論いけないピストンも無ければいけない、だから複働喞筒は小さな喞筒では面倒である因て之をやめて瓣を矢張二つにしてプランヂャル一つ用ゐた方が宜しい、そこで複働喞筒と同じき効用をなしながら複働喞筒でもないものがあるから其れを序に申します此處(第七圖)にあります是れには瓣が二つ共上向きに開いて筒が二つあります此處(a)が吸込み此處(b)が吐出し、水が昇るから此瓣が此處に行く其次に此方のバッケットは上に昇りますぢやに依て此處に行きます今度は是れが降つて是れが昇つて行きますと此處は水が素通りして仕舞ふ併ながら此方の瓣が閉ぢるから水がバッケットの下に付いて上る、上に居る水は押出されて仕舞ふ斯う云ふ工合になる是の喞筒には筒

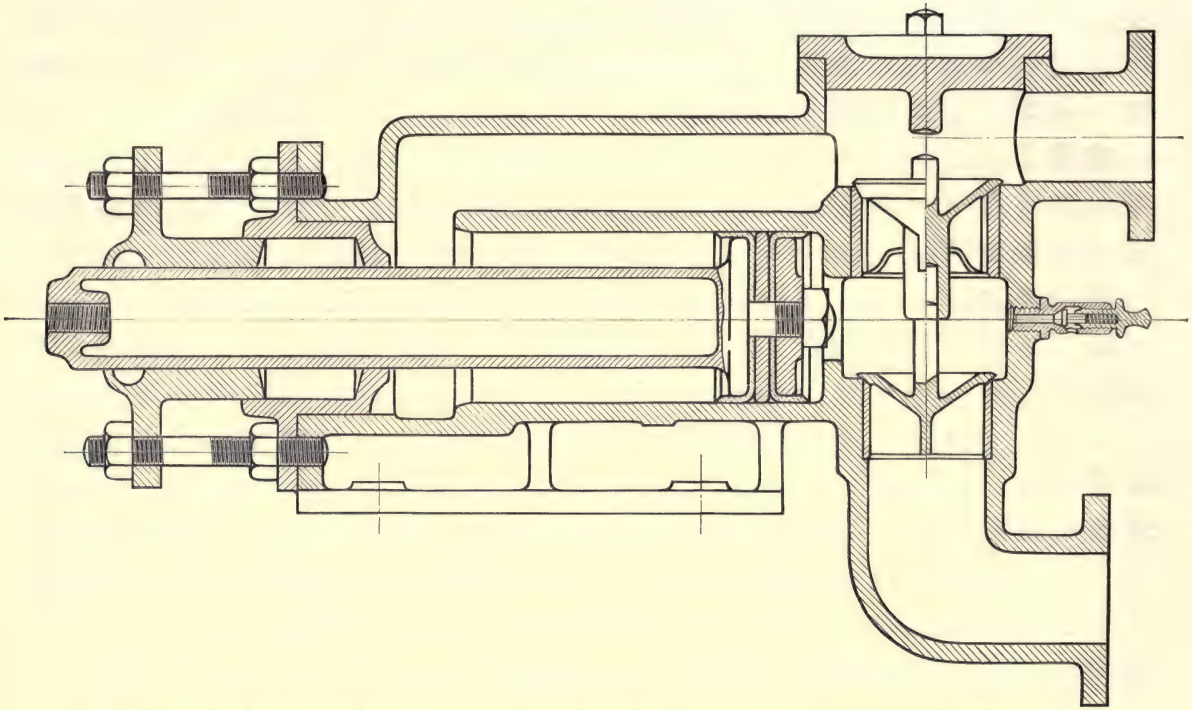
を直すことも出來ませう、然れども斯う云ふ簡單の蒸滊機關では單純なる普通の滑瓣一個のみを備へてある故に一方で蒸滊の遮斷を早くすれば外に不都合の件がおこる將又ピストン棒を太くすれば宜しいとは雖も摩擦が餘計になり且つ其大なる面積を有する棒が一往復毎に冷たき空氣に觸れて蒸滊の損失を生ずる、就ては運動の不同を整調するに最も容易な方法は相當なるはずみ輪を用るに在る、然るに普通出來合のドンキイに於てははずみの寸法不相當に小さいが常である、今此處に一例を擧てはずみ輪の大ならざるべからずと云ふことを申しませう、一つのドンキイがありまして八十磅の蒸滊壓力を持つて居る罐へ水を送ると假定します其プランヂャルの徑三吋卽ち其截斷面積は七平方吋衝程四吋半速度が一分時間五十七呎其れで回轉は六十八となります所で是れは小さな喞筒でありますから今はずみ輪の平均徑を假りに二十二吋半として速度の不同は一回轉中平均速度の十分の一までに達して宜しい卽ち平均速度の十分の一までは速くなつたり遲くなつたりして差支ないと假定す是に由て勘定致すとはずみ輪は九百九十六磅卽ち殆ど千磅なければならぬ是輪の周圍の截斷面積が五十四平方吋となる大變大きい者で

す即ち單働喞筒一個なれば最大速度から平均速度を減じたものゝ割合が二十一割四分となる三組の單働喞筒なれば其れが零割五分となる大變少ない、尤も此計算には空氣室内の空氣が働らかぬと假定いたしました勿論實際は空氣あるが爲め今申した割合より餘程平等になりませうが幾分か是の計算で速度の不同なる所が分ります

其れから更に一歩を進めて働力の方でどう云ふ關係があるかと考へて見ると次の如き結果が出ます、蒸滊罐などに水を送ります蒸滊喞筒と云ふのがありますが彼の類の中で小さな形のやつは複雑を避んが爲めに大抵單働喞筒に致します、こゝに（第五圖略す）例がありますプランヂャルと瓣が二つ付いて居るのみで卽ち例の往くときのみ仕事をして復りに仕事なしと云ふやつであります斯う云ふ蒸滊喞筒には回轉軸、クランク及びはずみ輪が付いて居る因て往動のとき緩くなり復動のとき急くなるべき所をば幾分か其はずみ輪で平等ならしむべし又は此處（第六圖略す）にありまする圖のやうにピストンの棒を大變太くして所謂トランクとなし蒸滊の這入る分量を一方に於てのみ少く致します或は出來るならばピストンの一方に於て蒸滊の遮斷を早くして幾分か不同の運轉

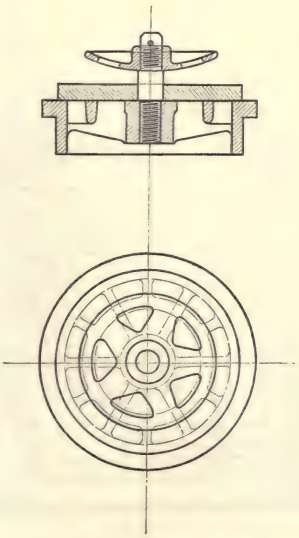
今假りに其往又は復のみに仕事をする喞筒を單働喞筒と譯して置きませう、さう云ふ喞筒に於ては平均速度と最大速度との割合が三、一四になります若又斯樣なる單働喞筒が二つなれば互に直角なる二つのクランクで運轉すると假定致します然るときは最大速度と平均速度との割合は二、二二となります又ダブル、アクチング、ポンプ卽ち譯して複働喞筒と云つて宜しう御座いませう此れか又は單働喞筒を二つ並べてクランクを百八十度に置た者か或は百二十度に置た者かどちらでも速度に對しては同じことで最大速度と平均速度とは一、五七斯う云ふ割合になる其れから單働喞筒を三つ並べまして其れを百二十度づゝの角度をなして居るクランクで運轉します是れをスリー、スロウ、ポンプ卽ち譯して三組喞筒と云ひませう是れになりますと云ふと最大速度と平均速度との割合は一、〇五の一に於けるが如くなります、其れから今度は單働喞筒を四つ或は複働喞筒を二つ並べてクランクを平等に配り仕事を致すと假定する、此場合では最大速度と平均速度とは一、一一の一に於けるが如し其れで一番前に申したものは三、一四に於ける一の如しで大變な差でありますす只今申した種類だけでは三組の喞筒の速度が最も平等になりま

第四圖



を申しますればプランヂャルは單に擂木のやうなもので只水の占めて居つた場所を無くすると云ふのが目的で其瓣は動かない其れに金屬製もあり革製もあり護謨製もあり種々御座ります金屬製で羽根のやうに動くのもあり昇るのもあり又球のやうな形のもあります其他にも隨分種類は澤山あるものです

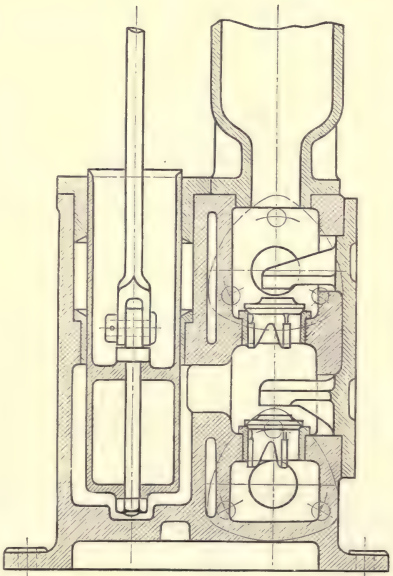
又一類の喞筒ではピストンとプランヂャルとを一所に結合した者もあります、其れは此處(第四圖)にあります是れがピストンでプランヂャルに固着して居る、凡て瓣なければピストンと言ひ瓣あればバッケットと言ふ、所で前に申しました種類の喞筒では總て皆往復運動の中の往又は復のみに水を放出する、往く時に水を放出すれば復りは空、復に水を放出すれば往くときは空、それでさう云ふ工合でありまするとと機械としての運轉上に大變な關係がありまする事は次の通りです



第三圖

ります

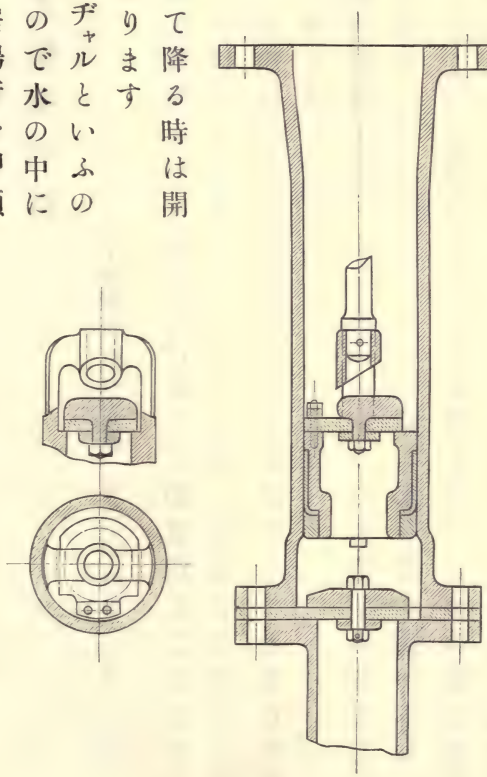
そこで兎に角どう云ふ喞筒でも瓣が少なくとも二つ無ければ仕事をする事が出来ない一方が開ひて居る時は他の者は閉て居る是れはどなたでも御存じでありますが今のバッケットと云ふ類の喞筒は動く部分に瓣を持つて居ると云ふ所が一種特別で其れからプランヂャル喞筒でありますと二つの瓣は動かない一定の位置に止つて居つて開閉すると云ふ類でありますそこで其バッケットの種類を申しますと云ふと護謨又は革を以て拵へた瓣を持つて居るのが澤山あります此處（第三圖）に書いてあるのは護謨又は革でありまず場合に依りましては金屬製の瓣を持つて居るのも隨分あります其れから今度はプランヂヤル喞筒の方



第二圖

內に密合しながら自由に往復運動をなすべき者で其れに瓣が付いて居ります一方に行くときには瓣が開き他の方に行くときには瓣が閉ぢます因て一方にのみ水を送り出す斯う云ふ類がバッケット喞筒であります(第一圖)此處に付いて居るのが所謂バッケットで昇るときは閉ぢて降る時は開く斯う云ふ譯であります

其れからプランヂャルといふのは擂木のやうなもので水の中に這入りまして水の居場所を押領する依てそれだけの水が他に行かなければならぬから其水が管を通つて流出する斯う云ふ類の喞筒であります此類は此處(第二圖を示す)にあ



第一圖

三。

汲揚及壓揚喞筒。

〔工學會誌 第百十九卷、明治二十四年十一月。(1891)〕

今日は「汲揚及壓揚喞筒」と云ふ題を掲げまして御話致すことになりましたが其主なるものばかりに就て申上ますと誠に簡短でござりますから少しく前置きに此類の喞筒一般に就て豫め御話致して其後に本題のことを申上げます事に致します

そこで先づ「汲揚及壓揚」と云ふこと是れは一つの名稱でありまして英語で申すと「汲揚」と云ふことはバッケット水杓(ひしゃく)即ち水を汲み揚げるもの「壓揚」と云ふことはプランヂャル衝入子即ち水を壓し揚げるもの此二者を併せ有するに因て斯う云ふ名を付けましたので有ります然るに此別々の名を持つて居る喞筒がありますからして其れのことから申上げます先づ第一番に汲揚喞筒即ち英語でバッケット喞筒と云ふのは一つの圓筒の

ば二、六實馬力を得

に從ひ遲速等しからず概して言へば水の高低大なれば速度亦大にして間〻格外なるものあり京都在高野村針金製造所の水車は徑二十尺にして一分時に十六回轉をなし居れり是れ周圍の速度一秒に十六、七尺に當る卽ち高低の損失四尺三寸なり

京阪地方に於て水車米ツキの馬力を調査せしに其平均數は東海道のものに比して大なる差違あり今一二例を擧げて之を證せんに近江國大津より半里内外藤尾村卽ち彼の疏水工事事務所のある處に數個の米ツキ水車あり其内杵の重さ六貫目杵の昇降九寸六分昇降數一分時に四十而して一臼に米一斗五升を容れて八時間にツキ終る是れ一石の米を一時間にツキアゲルには三、一實馬力を要する割合なり又京都三條孫橋に次の如き例あり杵の重さ七貫目杵の昇降八寸七分昇降數一分時に三十八、杵數十六本にて二十四時間に七石二斗をツキアゲル是れ一石の米を一時間にツキアゲルには三、一實馬力を要する割合なり此外京都市内の或る二個所に於ては一、九實馬力と二、七實馬力白川村に於て二、〇實馬力二、四實馬力高野村に於て三、二實馬力又攝津の尼ケ崎に蒸氣機械にて精米をなす所あり之は二、二實馬力を要する割合なり、以上八件を平均すれ

て水一馬力一時間に付き金六厘三毛を得是卽ち駿州より勢州に至る東海道各地水力の平均相場なり

三重縣四日市近傍及び京阪地方にては水車構造の細目稍々東海道のものと異なる所あるが如し例へば水受と半徑線とは幾分か相傾斜するもの京阪地方等に最多し此角度四十五度に達するもの往々あり又下射

車及低き胸射車にては全く側板を用ひざるものあり其形蒸氣船の外車に似たり又圖の如く胸射車の底板に於て水受毎に狹き通氣孔を備へたる者多し又上射車に在ては通氣孔を施すこと甚た不便なるに由り上渠の幅をば車の幅より狹くして水の進入と空氣の退出を自由ならしむるを善とす此方法を施せる者亦これあり特に攝津の住吉近傍に多し假に一例を擧げて之を言へば水車の幅を內法（うちのり）にて三尺六寸ありとすれば上渠の幅は水車と相接近する所に於て三尺二寸にして左右に各二寸の間隙あるが如し勿論上渠の幅は水車を距ること數間以上なれば何れの寸法にても通氣法には關係なし

京都近傍には純然たる日本風の水車頗る多くして其構造は上に述べたる如く少し進歩せし所ありと言ふを得べし水車の周圍の速度は場所

多くして一分時に四十五回より六十回まで又曰は一斗二升入り一斗三升三合入あり米ツキ馬力も大同小異にして今駿州三島邊より勢州四日市近在までの平均を取るに次の如し一石の米を一時間にツキアゲルには五、四七馬力を要す大津、京都、大阪、神戸等にて此馬力甚だ減少して三馬力前後の者多し尚ほ後に之を陳述すべし又嶋田には紡績所ありトルビンにて仕事を爲し居れり序に一言せんに水の高低十二尺五寸水量一秒時に三十立方尺餘ありて三十馬力を得ると云ふ是はフルネロン式の車にて某君の意匠に因り横須賀造船所に於て製作せし者なるが一分時に六七十回の廻轉を爲し平滑に紡績機械を運轉す玆に米ツキのことに面白き結果があり卽ち一俵に四斗入りの米を上白にツキアゲルには二升五合減少す並白にツキアゲルには一升五合減少す又一俵のツキ賃は並六錢五厘卽ち一石のツキ賃十六錢二厘五毛なり然るに一石をツキアゲルには五、四七實馬力を一時間要す因て之を計算すれば一實馬力一時間の價直は大凡三錢なり右に付水車及び米ツキ機械の機械的能率（efficiency of mechanism）を奇零七と假定すれば一指示馬力一時間に付二錢一厘なり又水車を起働機と見做すときは其能率は奇零三を超えざるべし因

用し居る者十を以て算ふべし此邊には急流多きが故に上射車を用ふる所稍々多し但し大抵皆底板を張りつめて水の入口に側面間隙を見ず故に空氣と水と混同してゴトゴトゴトの音斷ゆるなし水車の仕事は矢張米ツキと粉ヒキが重なれども其中烟草を刻む工場一個所あり其水車は徑一丈の胸射車にして水の高低五尺ばかり水量極て少し周圍の速度は一秒時に十三尺八寸あり此類の車にて無類なる速さと申すべし水車軸に鐵製齒車を施し其に小車を組み合せて回轉數を增加せしめ右小車の軸心にクランク數個を施し以て通常の烟草刻み機械の取手を運動せしむ又別に遊滑車をも刻機械一臺毎に備へたり斯くて一臺に付き其仕事は人力を用ふるときの三四倍に至ると云ふ大宮に撚糸場あり是れにも水車を用ふ其構造と使用法と兩つながら甚だ宜しからず

夫より駿州嶋田の宿を巡視せり此宿は田地六百五十町歩戸數千八百ばかり人口七千五百ばかりなるに水車の數は四十八あり習米ツキ用に供す其米は皆食用のみに供す此邊の水車は重に下射車にして其構造は前の各地方の者と異なることなし周圍の速度も略ぼ相似たり大抵一秒時に四尺乃至六尺なり米のツキ方も亦相似たり但し杵の昇降數は稍々

水受毎に狹き底板あり幅六寸軸心は一尺一寸六分徑の八角槻なり水面の高底二尺六寸水量一秒時に十立方尺六但し此水量は gauge notch にて測りたる者なり水車の回轉は一分時に五、七回依て周圍の速度は一秒に四、六尺なり

此工場の米ツキ杵の重さは六貫目乃至七貫目杵の昇降距離は一尺五分昇降數は一分時に三十八回なりき二斗入りの臼十二個ありて其米は二十四時間に一回半ツキアガル又一斗四升入りの臼十六個あり此は二十四時間に大抵二回ツキアガル因て一石の米を一時間にツキアゲルには五、一實馬力を要する割合なり前に申したる者と大なる相違なし此近邊數個所の米ツキを見るに杵は皆例の胴衝にして其重さは五貫以上八貫以下なり杵の昇降距離は九寸より一尺一寸位まで杵の昇降數は一分時に三十五回より五十回位まで臼の容積は一斗四升なるあり一斗五升なるあり又二斗なるあり其ツキアゲは二十四時間に一回以上二回位まで極稀れに二回半に及ぶものあり一石の米を一時間にツキアゲル爲め要する所の馬力は大抵四、五以上六、五位までとす七馬力に及ぶもの甚だ稀れにあるのみ今泉村を距る三里許の處に大宮と云ふ所あり水車を使

勾配を付け其より又三尺の間に五寸の勾配を付け此處で車へ水を落す尤も車へ落る處では水路の幅三尺位の者を二尺六寸位にしぼり縮めます水受と水受との距離(周圍にて測りたる)は平均八寸位なるが一尺餘のものも有ます水受の深さ(此深さ總て半徑線に沿て測る)は大抵一尺位又是より狹くすることも有ますが然るときには水返(かへし)(底板卽ち sole plate のことなり)が入用です此水返を付るは宜しからず水が車の胴を衝き推すのみにて其效能少なしと

右圈點を付せる所に稍々意味あり

余は是より進んで駿河國富士郡に到れり此一郡にて水車の數百拾五個舂臼の數一千二十四基、挽碓の數百十五基あり但し是は收稅課に於て舂臼三基以上を使用せる水車の統計に係るものとす本郡吉原在に今泉村なる有り水車數十個を使用し居るも皆下射車若しくは低き胸射車なり其業は例の如く米ツキ粉挽のみなり余は此村に於て大小、速度、高低、水量等稍や精しく測定せし所あり玆に之を陳へんに水車の徑一丈五尺五寸幅三尺(但し內側にて測る)水受の數四十枚水受の深さ一尺一寸五分水受と半徑線と互に傾斜せり其割合は半徑線一尺に付傾斜二寸五分なり

例の如く水車で米ツキと粉挽を業とする者頗る多し其内一個所にては杵の重さ五貫目、杵の昇降距離八寸強昇降の數は一分時に五十五回にして一臼に二斗入の米を二十四時間にツキアゲル是れ卽ち一石の米を一時間にツキアゲルには六馬力九を要する割合なり此水車の徑は二十尺幅は外側にて三尺二寸五分水受の深さ一尺、水受と水受との距離は一尺三寸軸心は一尺一寸の八角棰にして腕木は十六本あり皆二寸角の木なり車の兩側に四角棒にて方形匡を付けたり水受の板の厚さは六分、車の回轉數は一分時間に八回なりき水の高低は三尺五寸八分あり軸心は棰を用ゐ其他の部分は盡く松を用ひたり水車の徑二十尺にして一分時の回轉八なるが故に周圍の速度は一秒時に八尺四なり此水車には底板なくして水受の方向は半徑線に從へり沼津近傍に於て或る水車大工に就き水車の大小廣狹等を定むるの順序如何を質問せしにボンヤリしたる答にて一定の法則若くは仕樣方の如きは一として有るなし、云く水車の徑は成るべく大なるを好しとするが先づ水面高低の差を見て目分量にて極め車の幅も矢張目分量で極めるなり但し三尺位が廣いので有ませう車へ水を導くべき水路は大抵幅三尺位にして三間の間に於て三寸の

に之を用ふること極て稀なれば今夕の演說にも米春と挽粉の事甚だ多し諸君幸に之を諒せよ

相摸國の水車は之を後日に讓り余は先づ駿河國より說話すべし該國駿東郡のみにて水車現在の箇處百三ケ村にして米春と麥の粉挽に使用せる車は二百八十六個あり米ツキのみに使用せるもの九十三個あり卽ち總計三百七十九個なり是等の水車は盡く木造にして所謂胸射及下射車なり上射車は僅かに二三個に過ぎず車の徑は大抵二十五尺位より十三四尺迄を通例とし其幅は概して狹く三尺內外なり水受は平なる板にて造れるもの最も多し二枚の板を用ひ、へ字形に造りたるが如きは甚だ少し底板は或は用ひ或は用ひず今其一例を擧げんに沼津在三枚橋新田村の水車は米ツキ用に供し居るが其杵の重さ七貫目乃至八貫目あり其昇降距離は九寸強にして昇降の數は一分時に六十回、一臼に四斗入の米を二十四時間にてツキアゲルなり蓋し此杵は所謂胴衝なる者にして米ツキ仕事の馬力を計算するには最も都合よきものなり此例にては卽ち一時間に一石の米をツキアゲルには五、四七馬力(實用馬力)を要する割合なり又沼津より二十町許を距る處に香貫村と云へる村あり此村にては

二。

日本水車の說。

〔工學會誌 第七十一卷及び第七十六卷、明治二十年。(1887)〕

余は本年七月中旬帝國大學の命に因り水車取調の爲め東海道の各地方及ひ京都大阪神戸生野を巡回し八月下旬歸京せり今夕の演說は此巡回中に見聞せし所の日本水車及び水揚機械の構造使用法等に外ならず玆に豫め一言を要するあり他なし右の巡回は日數僅に四十餘日にして日本の東方一部分のみに止るが故に素より完全なる結果を得ること甚だ難し又日本固有の水車を使用し居る工場は西洋仕掛の工場と大に其趣を異にし何事を尋ても一として確然たる答辭を得る能はず已むを得ず質問者に於て自ら其寸法、速度及分量等を測らざるべからず爲に意外の時間を費せるは是れ亦其調査に不完全を來せる一原因なり且日本水車は主として舂臼と挽碓を運轉するの用に供するものにして他の工事

子桿取り付方の如くせり

側板は銑鐵板にて數個の扇面形に作り曲鍔を鑄付けてボルトと鉸釘を以て之を水受板に付着す扇面の内周には突出せる鍔あり底板を受るに便す臂は其徑通常二吋より二吋半なり斜なる結材は一吋四分之三より二吋とす底板は錬鐵にて作り其厚さ八分之一許にして鉸釘にて聯接するなり但通氣法を施したる車には底板を用ひざること多し

（不　完）

車を軸に取付るの法は工人の意見に從ひ其理大に異同あり其一法は鉄鐵臂を以て車を固結し之をして隔囘に引長力と壓搾力に抵抗せしむるなり又被牽棒(テンションロッド)數條を以て車を結約するは他の一法にして改良鐵製水車には專ら此法を適用す第一法にては臂は鉄鐵鑄物にして鉄鐵中心匡の凹處に嵌め込みボルトにて固むるなり或は錬鐵板の臂を以て中心匡に鉸釘するあり其法略ほ相似たり

所謂釣掛(サスペンションプリンシプル)の理に從ひ水車を造るときは輕き錬鐵棒にて軸より車を釣て足れり故に鐵棒に代るに鎖を以てするは道理に符するのみならず實地上亦頗る簡約なるものにして嘗て二三の水車に之を施せしことあり又荷車の輪に此法を用ひたる者あり然れども鎖は柔軟なるが故に多少不便なる所あり遂に廣く用ふるに至らず錬鐵の圓棒を至便なりとす舊來の釣掛構造法水車に在ては臂及び斜なる結材(ブレース)の端に螺旋を施し雌螺にて中心匡に取付るなり斯の如くすれば臂及び結材を緊着するに甚だ便なれども運轉中震搖を起すが爲めに其雌螺弛み釋け危害水車に及ぶことあるべし是其不利なり因てフエーヤベルン氏は臂の端を方形に作り雁鍵と栓とを用て中心匡なる凹處に緊着すること一に蒸滊機關の吸

$$D=0\cdot300\sqrt[3]{M} \quad \cdots\cdots \text{卯式}$$

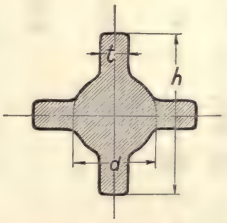
$$D=0\cdot250\sqrt[3]{M} \quad \cdots\cdots \text{辰式}$$

鍊鐵軸なれば

にて充分なり木製なれば銑鐵のものの二倍半の大さにて足る又方形軸なれば其方邊 $s=0\cdot94D$ 又内空の圓筒軸なれば戊式を用ふべし第四圖の如く肋骨を施す時は其總高さ h は徑 d の三倍にして厚さ t は徑の三分の一なるを常とす然る後此截面の權積を算して圓面其徑 D なる者の權積に矩合せしむれば

$$d=0\cdot576D \quad \cdots\cdots \text{巳式}$$

を得故に $h=1\cdot727D$ 及び $t=0\cdot192D$ なり



第四圖

Pr 及び M なる權積相比較して其一甚だ大なれば他の一は細微として棄去るを得べし卽ち扭拉權積大なれば丙丁式を用ひ枉屈權積大なれば卯辰式を用て可なり然れども右兩權積の差細小なれば須く

$$M+\sqrt{M+(Pr)}$$

を以て丙丁式の Pr に代用すべし

小部分のみ而して頸上壓力はW_0あるが爲めに增加すと雖も之れ亦少許のみ故に車軸は常に平直なる者と假定し其重心は甲乙の中央に位し兩頸上壓力の增加は各々W_0の一半と見做して可なり因て子、丑、寅點の枉屈權積（モーメント）は車軸の重力あるが爲めに增加すること左の如し

$$\frac{1}{2}W_0\frac{L_1(L-l_1)}{L} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \text{子式}$$

$$\frac{1}{2}W_0\frac{L_2(L-L_2)}{L} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \text{丑式}$$

$$\frac{1}{2}W_0\frac{L_3(L-L_3)}{L} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \text{寅式}$$

これを以て前段のM_1 M_2 M_3に加れば枉屈權積完全するなり乃ち軸の大小は子丑寅各點に於て之を定むるを得べし軸若し平直にして其徑一樣なるときは固より最大權積を用ふべし然れども枉屈權積は兩端に至るに隨ひ次第に減少すれば車軸の大さも亦左右に減少するを宜しとす車軸に肋骨を施し兩端に向ひ其高さを遞減するは之が爲なり

前術により頸上壓力を定め而して最大枉屈權積を算し然る後ち之を用て車軸の大を求むべし圓筒銑鐵なれば其徑D吋は

$$P=W_1+W_2+W_3-Q \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \text{庚式}$$

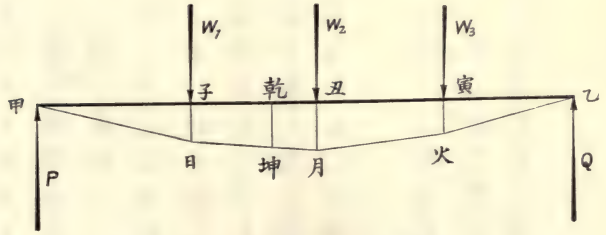
斯の如く頸の壓力を定め而して子丑寅諸點の枉屈權積（ベンディング、モーメント）を求むること左の如し

子點　$M_1=PL_1$ ……辛式

丑點　$M_2=PL_1+(P-W_1)(L_2-L_1)$ ……壬式

寅點　$M_3=PL_1+(P-W_1)(L_2-L_1)+\{P-(W_1+W_2)\}(L_3-L_2)$ 癸式

乃ち子丑寅諸點に於て右の權積に比例したる垂線子日、丑月、寅火を畫し直線を以て其端點甲日月火乙を連接すれば中間任何點乾の權積（モーメント）は其垂線乾坤と同比例をなすなり故に最大權積（モーメント）の生ずる所は必ず子丑寅なる力點の一に在り重力W_1 W W_3は臂を經て軸に達し子丑寅なる定點に在て作用を做す此外に車軸自ら重力W_0ありて甲乙全長に通じ其作用平等なり但之を以て車の重力Wに比する時は僅に一



第三圖

聯接するときに限るべし又一組の臂若くは一方の側板に先轉車を取付たるときは他の一組の臂はPrなる權積(力と距離の相乘をモーメントと名く今假に權積と譯す)の一部分のみを車軸に傳ふべし故に軸徑小なるを得臂二組ある時權積(モーメント)Prの一半は一組の臂より車軸に通じ車軸より他の一組に由て齒輪に達するなり臂三組ある時はPrの四分之二は中間の一組より四分之一は第三の一組より各々車軸を通じて第一組に達し乃ち齒輪に至るなり故に車軸の大小を算するにP若くは$\frac{1}{2}P$を以てPに代へ$H.P.$若くは$\frac{2}{3}H.P.$を以て$H.P.$に代ふべし又車の兩側に各齒輪を施し以て力を引用するときは車軸扭拉力を受ることなし故に軸徑を算するには枉屈作用の理に從ひ車の重力のみを論ずれば足る

車の重力に相當して車軸の大さを定むるには先つ兩頸上の壓力を知べし第三圖中甲乙なる車軸の長さをLとし子丑寅なる點に於て$W_1$$W_2$$W_3$の荷ありとし甲子、甲丑、甲寅の距離を$L_1$$L_2$$L_3$とし甲頸の壓力を$P$乙頸の壓力を$Q$とす然れば

$$Q=\frac{W_1L_1+W_2L_2+W_3L_3}{L} \quad \cdots\cdots\cdots\cdots\cdots\cdots \text{己式}$$

むること左の如し

$$D=0{\cdot}357\sqrt[4]{Pr}=6{\cdot}2\sqrt[3]{\frac{H.P.}{N}}\quad\text{時}\quad\cdots\cdots\cdots\cdots\quad\text{丙式}$$

Pは車周に働く所の力なり磅にて算すrは半車徑なり呎にて算す$H.P.$は馬力なりNは毎一分車の廻轉數なり

又錬鐵軸を用る時は左の如し

$$D=0{\cdot}297\sqrt[4]{Pr}=5{\cdot}16\sqrt[3]{\frac{H.P.}{N}}\quad\text{時}\quad\cdots\cdots\cdots\cdots\quad\text{丁式}$$

木製軸の徑は銑鐵の二倍なれば充分の強さあり、然れども實地の慣習にては三倍若くは四倍に及ぶ四角軸を用る時は其角邊$S=0{\cdot}94D$なるべし又内空の圓筒軸には其外徑

$$D_1=\frac{D}{\sqrt[4]{1-m^4}}\quad\text{時}\quad\cdots\cdots\cdots\cdots\quad\text{戊式}$$

mは外徑を以て内徑を除き得る所の商なり其實數は通常奇零六なり故に外徑$D_1=1{\cdot}05D$なり

以上丙丁戊等の式を用ふるには始めて力を受る所の先轉車水車軸に

頸の長さ通常其徑の一倍半とす

トレッドゴルド氏水車軸の徑を算する法は

$$D=\frac{1}{6}(LW)^{\frac{1}{3}} \quad \cdots\cdots\cdots\cdots\cdots\cdots \text{甲式}$$

Dは頸の徑吋にて算すLは其長さ亦吋にて算すWは頸にて支ふべき最大の重さなり磅にて算す又水荷を受る方に於て力を引用し小歯車にて水の重量を支ふるときにはWは車の重さ半分なり頸徑を算する別法あり左の如し

$$D=\frac{1}{25}\sqrt{W} \quad \cdots\cdots\cdots\cdots\cdots\cdots \text{乙式}$$

この式にては頸の長さは大約徑に等しきなり以上両式何れを用るとも前表の徑よりは稍小なるものを得

バイスバック氏書中軸頸の徑を算するの法あり甲式と同形なれども其の分母は九個に非ずして七個六三に當る又頸の長さは徑の一倍乃至一倍二五と云へり

水車の軸の大さを算するには其扭力に抗するの強さと其重力を支ふ強さを併せ考ふべし今扭拉力のみ存すると假定し圓形銑鐵軸の徑を定

Aは圓心に於て弧背に對する角の折半なりRは半圓徑なり故に車の外圍と底板との中間に一圓を畫し水の滿ちたる水受を橫過する所の弧背を取り右の法に從て其重心を求れば車中の水の重心を得るなり車軸全く水の壓力を受けざる樣になすには須く齒輪と齒車二組を備へこれを車の兩側に付すべし又一組のみを用ふるときは底板の裏面に接して斜なる被縮棒（ストラット）數條を施しこれをして車の一側の水力を他側に傳遞せしめ以て車胴扭拉するの患を除くを善とす

左の一表は車軸の頸（ジャルナル）の大小を示すものにて高蔽胸、低蔽胸何れに用ふるも可なり是れフエーヤベルン氏が四十年間の實行に出るものと云ふ

水車軸の頸大小表（頸の徑吋）

車徑呎／車幅呎	五	十	十五	二十	二十五	三十	四十	五十
十五	六	七	八	十	十一	十二	十三	十四
二十	七	八	九	十一	十二	十三	十四	十五
二十五	八	九	十	十一	十二	十三	十四	十五
三十	九	十	十一	十二	十三	十四	十五	十六
四十	十	十一	十二	十三	十四	十五	十六	十七
五十	十一	十二	十三	十四	十五	十六	十七	十八

$N=\frac{2\pi r}{d}=6\frac{r}{d}$　に略ゝ近しとす

竪水車の軸は重大なる銑鐵棒にて造り軸脚（ブラマブロック）にて之を受けボルトを以て車室の基礎に取り付るなり車の重力は必ず車軸にて之を支ふ車の力を傳遞するの法は三樣あり其一は車軸よりす其二は直ちに車の周邊よりす其三は車の臂棒（アーム）に傳遞具を施し車軸と周邊との中間よりす第一法を用れば車軸は車竝に水の重力を受るのみならず又自ら扭力に抗せざるを得ず是れ水力は車邊より臂に達し臂より車軸に達すればなり車軸には通常肋骨（リブ）を鑄付け以て車軸廻轉する毎に引長力と壓搾力とに抵抗し得る樣に割合し置なり第二法には車の周邊に齒輪を施し之を齒車に組み合してこれより力を引用するなりこの法を用れば車軸の強さは僅に車の重力を支るのみにて足れり其齒車の接點（ピッチポイント）は車中の水の重心を通し車軸に並行せる垂平面中に在るを要す圓弧背（サーキラーアーク）の重心と圓心との距離は半徑と弦の相乘を置きて弧背の長さを以て之を除き得る所の商に等し

則ち

$$\frac{R\times(2R\sin A)}{2RA}\quad\text{卽ち}\quad\frac{R\sin A}{A}\quad\text{なり}$$

より小なる充塞率（コエフィシエント オブ フィル）eを以て右の容積に乗じ然る後之をQに適合すべし

$$Q = eubd\left(1 - \frac{d}{2r}\right)$$

ランキン氏の書に據れば此乗率eは三分の二なりバイスバック氏の書に據れば四分之一乃至三分之一にして車徑大なる時は五分之一に及ぶと云へり上射水車には胸渠を施さゞるを常とす故に充塞率小なるを善とす之を定むるは大に意匠者の特見によると雖も蓋し三分之一乃至三分之二を取り大過不及あらざるべし車の幅を得る式左の如し

$$b = \frac{Q}{eud\left(1 - \frac{d}{2r}\right)}$$

水受の數を增加すれば各水受中の水量減ずべし各水受中の水量減ずれば水、水受に留駐すること久し故に凡そ水受の數を增加すれば能力率亦增進すべし但其水受の數は定度あり妄に多くすべからざれども薄き鐵板にて造るときは其數木製のものよりは稍々多きも妨なかるべし今車の外圍にて測りたる水受の距離と側板の深さと相等しからしむれば水受の數を得る式左の如し

故に第貳圖BODなる扇面形の面積は$\frac{1}{2}ur$平方呎なり又內圍の速度は$u\frac{r-d}{r}$呎なり故にEOFなる扇面形の面積は$\frac{1}{2}u\frac{r-d}{r}(r-d)$

卽ち$\frac{1}{2}u\left(r-2d+\frac{d^2}{r}\right)$平方呎なり故に$BEFD$なる面積は

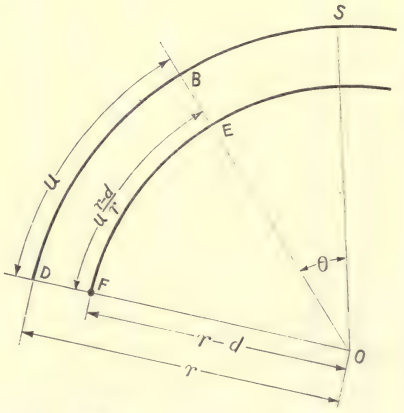
$$\frac{1}{2}ur-\frac{1}{2}u\left(r-2d+\frac{d^2}{r}\right)$$ 平方呎

なり卽ち $ud\left(1-\frac{d}{2r}\right)$ 平方呎

なりこれに車の幅bを乘じて毎一秒水に面する所の容積を得

$$ubd\left(1-\frac{d}{2r}\right)$$ 立方呎

なりこの容積は毎一秒の水量Qと等しくす可からず其故三あり一には各水受の容積は水受二枚の間の容積より少なければ也二には水受に水を充滿せしむれば其流出すること早くして損あり三には水受の容積Qに等しければ水の流入自在ならず其若干部分は迸出して用をなさず故に先づ一個



第二圖

を得又半車徑の略近商を求め

$$r=\frac{H\{1-0\cdot000047\,(kw)^2H\}}{1+\cos\theta}\quad 呎$$

を得因て車邊の速度は毎一秒

$$u=\frac{\pi nr}{30}=0\cdot1047\,nr\quad 呎$$

又水の流入速度は毎一秒

$$v=0\cdot1047\,kw\quad 呎$$

水車側板（Crown or Shrouding）は其深さ小なるを善とす何となれば車徑若干呎にして側板淺ければ水其作用を做すに當り挺枝長きときは得にして失にあらず側板深ければ挺枝短きに失すればなり凡そ側板の深さは〇、九呎より一呎四分之三許なれども大約一呎より一呎四分之一にして過不及なかるべし今dを側板の深さとし呎にて算しbを車の幅とし亦呎にて算すれば側板と底板（Sole Plate）とにて圍みたる圓筒形なる容隙の横截面積はbd平方呎なり而して彼の外圍の速度は毎一秒u呎なるが

如し

$$v=\frac{H-1\cdot1\frac{v^2}{2g}}{1+\cos\theta},$$

若くは

$$v=\frac{H-k^2\,1\cdot1\frac{u^2}{2g}}{1+\cos\theta}$$

此式にて車徑を算するには車邊の速度uを知るを要す然れども實際の場合にては毎一分時の車の廻轉數nを前定して然る後車徑及び各速度を定めんと欲する事多くあり今これに適する式を求むべし、何となれば

$$u=\frac{2\pi rn}{60}$$

なるが故に

$$(1+\cos\theta)\,v=H-1\cdot1\,k^2\left(\frac{\pi rn}{30}\right)^2\frac{1}{2g}$$

卽ち

$$(1+\cos\theta)\,v=H-0\cdot000187\,(krv)^2$$

を得この二次方程式を解き半車徑を求め

$$r=\frac{\sqrt{0\cdot000748\,H(kn)^2+(1+\cos\theta)^2}-(1+\cos\theta)}{0\cdot000374\,(kn)^2}$$ 呎

貳呎半に下る可らず此車邊速度uを定むれば流入速度vはkuに等しき故に直ちに之を算し得べし今rを半徑とし呎にて算すれば左式を得

$$H=(1+f)\frac{v^2}{2g}+r(1+\cos\theta)$$

此式中Hは水の總計高さ$(1+f)\frac{v^2}{2g}$は速度に由來する高さ、$r(1+\cos\theta)$は現存高さなり竪轉水車にては壓力に由來する高さなし故に總計高さは速度に由來する高さと現存高さの和に等しきなり、θは車頂と水の始て流れ入る點とに半徑線を畫し其間に介する角度なり水の流入自由にして迸出せざるを欲せば須く通氣法を備ふべし水受に通氣法を設けたるは蓋しフエーヤベルン氏に創る上射水車にては水の流入は車の頂點に近きが故に通氣法を施す事不便なれども水流入するの點車頂より三十度に下れば此法亦施し得べし卽ちθなる角三十度以上なれば則ち可なり此れ上渠水面の高低變化（ウアドヽース）一呎左右に過ぎざるものに適す其高低變化これより大なるときは須く蔽胸水車を用ひθなる角度を增て六十度より九十度にもなすべし然れば水扉（スルース）の位置を適應整理すること亦甚だ便なりfは抵抗率にして大抵高さの一割に當る因て半車徑を得る式左の

より左式を得

$$\frac{u}{v}=\frac{\sin(\beta-\alpha)}{\sin\beta}$$

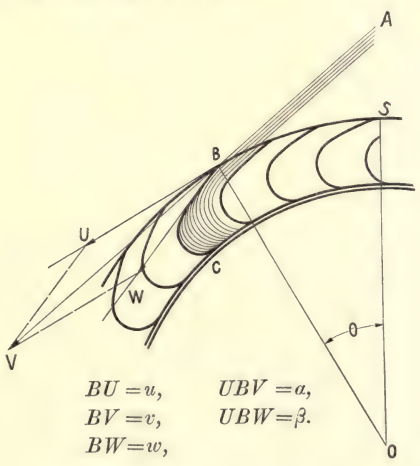
及び

$$\frac{w}{v}=\frac{\sin\alpha}{\sin\beta}$$

流入水の速度と車邊速度の比數 $v\div u=k$ は一、五より二に至る、β は水受の外端に觸るゝ線の方向と車邊に觸るゝ線の間に在るものにして之を流入角度（エントランス アングル）と云ふ車徑大にして速度緩なるときは二十度許、車徑小にして速度急なるときは四十度に及ぶ β 並に k を定むれば α なる進來角度（アングル オブ アプローチ）は左式にて算し得べし

$$\alpha=\beta-\sin^{-1}\left(\frac{1}{k}\sin\beta\right)$$

上射水車の外圍速度は緩なるを善とす車徑大なるときは毎一秒十呎に至る者あれども中間車徑には五呎左右なり又極小車徑と雖も其速度



第　一　圖

りとす何となれば水車の馬力八十に過ぐるときは重くして不便なればなり

能力の損失を減少せんが爲めには水始て水受に入る時衝撃なからしむること緊要なり故に水受の外端は必ず尖鋭に造るべし又第一圖中Oは車の中心OSは竪半徑、Bは水始て車に入るの點BUは車邊に觸るゝ線にして外圍の速度なり之をuとすBWは水受の外端に觸るゝ線にして水の比速度なり之をwとす、流入水の速度をvとす速度は皆毎一秒呎にて算すABはBCなる水受に入る所の水流にして之が衝撃を防ぐに其方向は次の法に從ふべしBVなる水流の速度を二に分解し其一部は外圍の速度uと等しく他の一部は水受の外端BWと方向を同くすべし此BWなる方向及び半徑OBに直角なる外圍の速度uを既知と定む然らばvなる流入速度も亦既知なるが故に其方向を得んにはUよりBWに並行して一線を畫しBを中心としてvを半徑として弧背を畫しVなる點に於て右の並行線と相交はらしむ則ち其BVは卽ち求る所の水流方向なり又BUに並行してVより一線を畫しBWを得是れ卽ち車に對したる水の比速度なり今UBV角をαとしUBW角をβとすれば算法に

イデン等の諸氏相繼で之を改良せり世上用ふる所の水力機械中トルビン水車其尤たるに至れり抑もトルビン水車は其運行快疾其車輕小にして構造の費用亦寡く而して能力率は却て甚だ大なれば昔時の上射車或は下射車とは比すべきに非ず近來合衆國に於ては所謂亞米利加風トルビンとて内流にして且つ下流なるもの特に行はれ其構造日に改良して其能力率八割に及ばざれば忽視せらる往々九割以上の能力率を得る事ありと是れ昨年十一月米國機械學士會に於て其會長サルストン氏が演説中に言へる所にして亞米利加トルビンの善美にして能力率の大なる事以て視るべし今前表にはトルビン水車の能力率を假に七割乃至八割とす

上射水車

凡て上射水車を用ふる所には水の高さ八呎乃至五十呎にして其水量は毎一秒三立方呎乃至二十五立方呎を常とす其最少高さと最少水量にては三馬力乃至五馬力を得べく又高さ水量幷に大なるときは百三十馬力にも及ぶべし然れども斯の如き場合にては水車二個を用ふるを利あ

$$Q=\frac{550H.P.}{u\,w\,H}\quad 又は\quad H=\frac{550H.P.}{u\,w\,Q}$$

*H.P.*は水車の馬力wは一立方呎水の重量六十二磅四二五、uは水車の能力率にして水車の種類、構造の美悪、速度の適否等に因て同じからざれども其概畧は左の如し

下射車、直なる水受（バックワット）のもの	○、三三乃至○、四
低蔽胸車（ローブレスト） 高蔽胸車（ハイブレスト） 上射車	○、五　乃至○、七
ポンセレツト氏下射車	○、五二乃至○、六八
改良鐵造高蔽胸車	○、六五乃至○、七五
トルビン水車	○、七　乃至○、八

竪轉水車はフエーヤベルン氏の改良鐵造蔽胸車幷にポンセレツト氏の下射車を最も利ありとす但其運行緩漫其車重大にして製造の費用甚だ大なり而して其能力率は七割五分に及ぶもの罕なり五十年前フアルニーロン氏トルビン水車の構造法始て行はれフオンテン、ジヨンバル、ボ

ルビン水車と名つくトルビンとは蓋し渦旋の意にして水の運行形狀に義を取るなり竪轉水車は車輪下垂にして竪轉し車軸は平置す是所謂通常水車なり今先づ竪轉水車に付て言ふべし竪轉水車に上射下射及び(ブレスト)胸(ブレスト)の三類あり第一類は水の流れ入る處車頂近傍に在り其作用は重力を以て主とす第二類は水車邊に接行して下部より流れ入り其作用は衝撃を以て主とす第三類は水を車腹に受け水の作用は重力と衝撃とを兼ぬ而して圓弧形の胸渠(ブレスト)あり車邊に密接し以て水の溢流を防ぐ因て蔽胸水車の名あるなり

凡て水力機械の意匠をなすに必ず左の二項を知るを要す

第一　毎一秒立方呎水量これをQと名づく

第二　水の總計高さ呎にて之を算しHと名づく

大なる池若くは河流の水を引き用ひて若干馬力の水力機械を造らんと欲する事多くあり或は又水量若干にして高さを上下するを得る事あり斯の如き場合にては相當の水量Q又は高さHを測算する事第一要件にして左式を以てこれを得べし

卽ち

$$h+\frac{p}{w}+\frac{V^2}{2g}=H \quad \cdots\cdots (4)$$

なる式中左邊の三項は大小有無の變化あれども右邊のH卽ち三類の和は定數にして變化なきなり此Hを名けて總計高さと云ふ

水の能力は以上述べたる如く三類あれども其計算には運動及び壓力を高さに變換して用ふるを至便なりとす總計高さH呎を置き之に毎一秒の立方呎量を乘じ又之に毎立方呎水の重さ六十二磅四二五を乘ずれば其得數$62{\cdot}425\,QH$呎磅は毎一秒の能力なり五百五十呎磅にて之を除し

$$\frac{QH}{8{\cdot}81} \quad \cdots\cdots (5)$$

を得是水に存する所の馬力なり之に水力機械の能力率を乘ずれば其機械の馬力を得るなり

水力機械を大別して水車と水壓機械との二類とす水車は必ず車輪車軸を有し水力を受け回轉して作用をなすの機なり水壓機械は水筒吸子を有し水の壓力を受け往復運轉して作用を做すの機なり水車に竪轉橫轉の別あり橫轉水車は車輪平置にして橫轉し車軸は直立す或は之をト

なり其gは地球引力の加速率にして毎一秒毎一秒三十二呎二なり此式中$\frac{V^2}{2g}$を名つけて速度に由來する高さと云ふ即ち物を墜下してVなる速度を得せしむるには其高さ$\frac{V^2}{2g}$呎を要すればなり

現存高さなく又運動なきも壓力あるときは必ず能力あるなり今pを毎平方呎の壓力とし磅にて之を算すれば水の能力は毎一秒に

$$pQ \text{ 呎磅} \quad \cdots\cdots\cdots\cdots \quad (3)$$

なり此式を$wQ\frac{p}{w}$斯の如く書して第一式のwQhと比較するときは$\frac{p}{w}$なる因數とhなる因數と相適應す即ちpなる壓力あるときは$\frac{p}{w}$なる高さあるときと其能力の量は相等きなり故に$\frac{p}{w}$を名づけて壓力に由來する高さと云ふ

水は唯能力一類のみを有することあり或は二類を有することあり或は三類並に存することあり或は此類漸次に彼類に變ずることあり而して水其變類の時に於て摩擦力を受けず又外部より抵抗力を受けざる場合にては三類能力の和は變ぜざるなり即ち算式にて之を言へば

$$wQh + wQ\frac{V^2}{2g} + pQ = wQH$$

圓を累ぬると雖もこれなきに等し水理に於けるも亦同一なり千萬立方呎の水ありと雖も高さなければ虱すら壓殺する能はず今三呎（フート）の高さの處に二磅の水あれば其水の能力は六呎磅なり反復して之を言へば二磅の水をして三呎の高さより墜下せしむる時は六呎磅の作爲（ウオルク）を得るを謂ふなり之を概言すれば h 呎の高さに於て Q 立方呎の水ありと假定すれば其能力は

$$wQh \text{ 呎磅} \quad \cdots\cdots (1)$$

なりと云ふことにて其 w は一立方呎の水の重さ卽ち六十二磅四二五なれども實用の計算には六十二磅半を用ひて可なり其 h は水の直下する高さにして高低測量により尺度を用ひて測り得べきものなり故に之を現存高さと名つく高さに三類あり曰く現存高さ曰く壓力（アクチユアルハイト）に由來する高さ曰く速度に由來する高さ

現存高さなきも運動あれば必ず能力あり是れ唯く水のみならず諸物體皆然り今 V を水の運動の速度とし毎一秒を以て之を算す然るときは其能力は

$$\frac{1}{2}\frac{wQ}{g}V^2 \text{ 呎磅} \quad \cdots\cdots (2)$$

ものあらず此れ英國を特に然りとなす大學士ランキン氏蒸氣機械書中に水車の理を論ぜり然れどもトルビン水車に至ては其理論に誤謬あり盡く信ずべけんやフエーヤベルン氏の書中記する所は一に同氏の經驗に出るが故に的切にして信ずべしと雖も水車の大小長短強弱遲速等を定むる所以の法を擧げず其他數部の書ありと雖も精粗詳略其中を得ず或は理論に過き或は實驗のみに偏す惜むべきなり余嘗て諸書を參考し傍ら愚見を交へて水車意匠法を纂め以て自ら閲覽に便にせり今爰に水力の理を總論し水力機械の區別を擧げ然る後水車の意匠法を言ふべし

水理を言ふには先づ能く其學術に用ふる普通の語を理會するを要す凡そ實理を論ずるには一字一句と雖も皆一定の義を存し決して曖昧妄意あるべからず席上の談話或は文學上の語類と大に差違あるなり今水理上の語を掲出して之れが解をなすべし

高さ　水理學にて所謂高さなるものは能力（エナルヂー）の強弱變化を測るの度若くは率にして猶ほ熱體の溫度の如し猶ほ電氣のポテンシヤルのごとし猶ほ銀行預け金の利息割のごとし熱にして溫度なければ能力なし電氣にしてポテンシヤルなければ能力なし銀行預け金に利息なければ千萬

媒介を生み又善く之を使用するの人を謂ふ故に本原動作者を造るは機械者に於ては最も重要の事たり近來電氣學日に益〻進むが故に機械工事は早晩一變すべしと雖も方今に在て之を論ずれば則ち本原動作者中實用に至便なるは蒸氣機械にして其理と構造とは大率講究して至らざる所なし實に精を窮め巧を極むと謂ふべし瀛機に亞て便なるものは蓋し水車なり水車は何れの國にても上代より既にこれあり但其用甚だ狹く或は田地に水を灌ぎ或は穀物を磨せしのみ結構極めて粗にして且つ拙なりき近世實學行はれてより其利用稍多く其理を講ずる者往々これあり其構造も亦頗る改良せり然るに蒸氣機械は陸地、水上、定所、動位用ふべからざるはなく其力の大小強弱欲する所に從ふ便宜これに若くものなし是を以て學者汲々これを講究す水車は唯燃料を要せざるを以て一大利となすと雖も水のある處にあらざれば用ふべからず其力は水量と高さに因り限ありて增益するを得ず則ち水力は蒸氣力の便なるに若ざるなり是れ水力は有用にして貴重すべきものなれども意を此に留むるもの少なき所以なるか河流瀑布の力を用ひざるは數萬斤の石炭を將て水に投棄すると一般にして惜むべきなり又水力を論ずるの書亦完備する

一。

水車意匠法。

〔工學叢誌（後に工學會誌と改稱す。）第十五、第十六、第十九卷、明治十六年。（1883）〕

蒸氣機械、水車、風車、力役夫、牛馬等は皆力作の業を做すものにして英語に之をプライム、ムーバルと云ふ譯すれば卽ち本原動作者と云ふ義なり往昔は人唯々日常必需の物を得れば自ら足れりとす故に工事極めて少なく作爲（ウオルク）を費す事亦從て少なし近世に及び世態大に變じ形より而下の學稍開け創意の事卽ち所謂る新發明なるもの相繼て起り千百を以て算ふ是に於て人生の需用亦千百之に從て生せり蒸氣機械の如きは其需用を得せしむるの媒介なり之に由りて作爲（ウオルク）を得る所以のものは一に石炭の能力（エネルジー）に原つくなり元來諸般の機械なるものは自ら能力を有するものに非ず又之を增損するものに非ず則ち專ら能力を傳遞し若くは之を適應するに便にする所以の媒介なり而して機械師或は機械學者とは好き

正誤

頁	行	誤	正
二二	一一	W_1 W_3	W_1 W_2 W_3
三二	一三	習米ツキ	特米ツキ
四四	七	はすみの寸法	はすみ輪の寸法
四九	一四	致しますと云ふ	致しますと云ふと
五〇	一	$2\cdot66\times15\,h$	$2\cdot66\times15\,h$
五六	三	第二百二十二巻	第百二十二巻
同	一〇	日本辭書	日本辭書
七四	七	輸送すべき管	輸送すべき管
七六	一二	噛き出す所の	吐き出す所の
九八	一一	適用せられるる	適用せらるる
一〇一	九	$f_b=\frac{yM}{I}=\frac{6W}{bh^2}$	$f_b=\frac{yM}{I}=\frac{6M}{bh^2}$
一〇一	一〇	$=\frac{6\times\cdot484\times W}{35\times4^3}$	$=\frac{6\times\cdot484\times W}{35\times4^2}$
一〇五	欄外	明治三十六年	明治三十九年
一〇六	同		
一〇七	同		
一一〇	一三	此符	此符號

八、圍爐裡の通氣法……六四
〔工學會誌　第二百五卷、明治三十二年二月。(1899)〕
九、實用數學摘要……七九
〔文部省實業學務局へ報告、明治三十五年十月。(1902)〕
十、木材の剪斷試驗報告……八六
震災豫防調查會委員　工學博士　井口在屋　合著
同　囑託　工學士　田中不二
〔震災豫防調查會報告、第五十五號、明治三十九年九月三十日。(1906)〕
十一、工業的數學……一〇八
〔全國工業學校徒弟學校長會議に於て講演。文部省實業學務局編、
全國工業學校徒弟學校長會議要項　所載、明治四十四年十二月。(1911)〕

五、材料强弱試驗報告……五八
震災豫防調查會委員　井口在屋
同　工學博士　眞野文二　合著
同　同　田邊朔郎
〔震災豫防調查會報告　第二號、第百五十五頁より第百六十二頁まで。　明治二十七年八月二十五日。（1894）〕

六、材料强弱試驗　第二回成績……五九
震災豫防調查會委員　井口在屋
同　工學博士　眞野文二　合著
同　同　田邊朔郎
〔震災豫防調查會報告　第三號、第十五頁より第二十六頁まで。　明治二十八年六月八日。（1895）〕

七、術語撰定の要旨……六〇
〔明治三十一年七月機械學會正員會に於て述ぶ。（1898）機械工學術語集、第一輯。明治三十四年十月。〕

目次

一、水車意匠法（デザイン）……一頁
〔工學叢誌（後に工學會誌と改稱す。）第十五、第十六、第十九卷、明治十六年。（1883）〕

二、日本水車の說……二七
〔工學會誌　第七十一卷及び第七十六卷、明治二十年。（1887）〕

三、汲揚及壓揚喞筒……三七
〔工學會誌　第百十九卷、明治二十四年十一月。（1891）〕

四、はやさ、速度……五六
〔工學會誌　第百二十二卷、明治二十五年二月。（1892）〕

を共にせんと欲し、先生の許諾を得て、其論文數十篇を集輯し、且つ之を刊行し、名けて井口集と曰ふ。集中收むる所は、先生の論文の主要なるものにして、中若干は未だ世に發表せられざりしもの、何れも不朽の文字、必須の研究にして、眞に先生の學德の記念とするに足る。大正二年二月版成る。乃ち本書刊行の由來を記して以て序となす。

大正二年三月

井口教授在職二十五年祝賀會

井口集編纂委員

序

教授井口在屋先生、職に東京帝國大學工科大學に在ること既に二十有五年、一意育英の任に當りて倦まず、又學術の研鑽に從事して厭はざること、終始一日の如し。其薫陶を受けて有爲の技術家たり、教育家たるもの、實に千百を以て數ふべく、其研究に係る論文及び教育に關する識見の、屢ば學會の講演、又は學術誌上に於て公表せられ、以て斯學の進歩に貢献せられたるもの甚だ多しとなす。茲に門人相議り、先生の知人の賛助を得て、水力實驗用渦巻喞筒裝置を東京帝國大學に寄贈し、又微品を先生に呈し、以て先生が在職二十五年の記念となし、且つ聊か祝賀の意を表せり。而して更に先生の偉績を世に傳へて、衆と其慶

東京帝國大學工科大學教授
海軍大學校教官
震災豫防調査會委員
臨時軍用氣球研究會委員
工學會會員、機械學會正員、造船協會正員

帝國學士院會員
工學博士、工學士　井口在屋（ねのくち ありや）

工學論文集

井口教授在職二十五年祝賀會　編纂

井口集

UNIVERSITY OF CALIFORNIA LIBRARY
Los Angeles
This book is DUE on the last date stamped below.

Form L9–30*m*-11,'58(8268s4)444

*TJ Inokuty -
7 Technical papers.
I58t

UC SOUTHERN REGIONAL LIBRARY FACILITY
D 000 703 115 6

*TJ
7
I58t